On the Tail Probability for Discounted Sums of Heavy-tailed Losses

Roger J.A. Laeven

University of Amsterdam www.fee.uva.nl/ke/laeven

Winter School on Mathematical Finance January 24, 2006

Main references

This talk is mainly based on

[1] Laeven, Roger J.A., Marc J. Goovaerts & Tom Hoedemakers (2005). "Some asymptotic results for sums of dependent random variables, with actuarial applications," *Insurance: Mathematics and Economics* 37, 154-172.

[2] Goovaerts, Marc J., Rob Kaas, Roger J.A. Laeven, Qihe Tang & Raluca Vernic (2005). "The tail probability of discounted sums of Pareto-like losses in insurance," *Scandinavian Actuarial Journal* 6, 446-461.

Outline

- The general problem;
- Classes of heavy-tailed distributions;
- The main results;
- Examples.

The general problem

Consider the randomly weighted sum $\sum_{i=1}^{n} \theta_i X_i$, with

- $\{X_n, n = 1, 2, ...\}$ a sequence of i.i.d. r.v.'s;
- {θ_n, n = 1, 2, ...} a sequence of non-negative dependent r.v.'s;
- the sequences $\{X_n, n = 1, 2, ...\}$ and $\{\theta_n, n = 1, 2, ...\}$ being independent.

We want to investigate its tail probability and functionals (risk measures) thereof.

The general problem: interpretation

- X_n : represents the net loss or payoff of an insurance or financial product (or portfolio, line of business, conglomerate,...) in (development) year n.
 - Is assumed to be independent across time.
 - In insurance, typically heavy-tailed.
- θ_n : represents the stochastic discount factor for year n.
 - Case 1: $\theta_n = Y_1 \cdots Y_n$, with $\{Y_n, n = 1, 2, \ldots\}$ a sequence of non-negative i.i.d. r.v.'s.
 - Case 2: no assumption on the dependence structure.
 Includes e.g., GARCH models.

The general problem: possible solutions

- Monte Carlo simulation;
- Easy-computable bounds or approximations à la Roger & Shi (1995);
- Asymptotics.

Classes of heavy-tailed distributions [1]

• Class S:

$$\lim_{x \to +\infty} \overline{F^{*n}}(x) / \overline{F}(x) = n,$$

for any (or equivalently, for some) $n \ge 2$.

• Class \mathcal{L} :

$$\lim_{x \to +\infty} \overline{F}(x+y) / \overline{F}(x) = 1,$$

for any real number y (or equivalently, for y = 1).

Classes of heavy-tailed distributions [2]

• Class \mathcal{D} :

$$\limsup_{x \to +\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < +\infty,$$

for any 0 < y < 1 (or equivalently for some 0 < y < 1).

• $\mathcal{D} \cap \mathcal{L} \subset \mathcal{S} \subset \mathcal{L}$; see e.g., Embrechts, Klüppelberg & Mikosch (1997).

Classes of heavy-tailed distributions [3]

• Class $\mathcal{R}_{-\alpha}$:

$$\lim_{x \to +\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha},$$

for any y > 0.

• Class $\mathcal{R}_{-\infty}$:

$$\lim_{x \to +\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = \begin{cases} 0, & y > 1; \\ +\infty, & 0 < y < 1. \end{cases}$$

Asymptotic results [1]

Let

• $\{Y_n, n = 1, 2, ...\}$ i.i.d. supported on $(0, +\infty)$;

•
$$Z_n := Y_1 Y_2 \cdots Y_n;$$

•
$$0 < a_n < +\infty, n = 1, 2, ...$$

If $F_Y \in S \cap \mathcal{R}_{-\infty}$, then it holds for each n = 1, 2, ... that

$$\mathbb{P}\left(\sum_{i=1}^{n} a_i Z_i > x\right) \sim \sum_{i=1}^{n} \mathbb{P}\left(a_i Z_i > x\right).$$

Asymptotic results [2]

Let

• $\{X_n, n = 1, 2, ...\}$ i.i.d. supported on $(-\infty, +\infty)$.

If $F_X \in \mathcal{D} \cap \mathcal{L}$ and $F_Y \in \mathcal{R}_{-\infty}$, then it holds for each n = 1, 2, ... that

$$\mathbb{P}\left(\sum_{i=1}^{n} (a_i + X_i)Z_i > x\right) \sim \sum_{i=1}^{n} \mathbb{P}\left((a_i + X)Z_i > x\right)$$

and that

$$\mathbb{P}\left(\sum_{i=1}^n (a_i X_i) Z_i > x\right) \sim \sum_{i=1}^n \mathbb{P}\left((a_i X) Z_i > x\right).$$

Asymptotic results [3]

If X and Y follow a lognormal law with $\sigma_Y < \sigma_X$, then it holds for each n = 1, 2, ... that

$$\mathbb{P}\left(\sum_{i=1}^{n} (a_i + X_i)Z_i > x\right) \sim \sum_{i=1}^{n} \mathbb{P}\left((a_i + X)Z_i > x\right)$$

and that

$$\mathbb{P}\left(\sum_{i=1}^n (a_i X_i) Z_i > x\right) \sim \sum_{i=1}^n \mathbb{P}\left((a_i X) Z_i > x\right).$$

Asymptotic results [4]

Let

• $\{\theta_n, n = 1, 2, ...\}$ non-negative and **dependent**.

If $F_X \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$ and there exists some $\delta > 0$ such that $\mathbb{E}[\theta_i^{\alpha+\delta}] < +\infty$ for each $1 \le i \le n$, then it holds for each $n = 1, 2, \ldots$ that

$$\mathbb{P}\left(\sum_{i=1}^{n} \theta_{i} X_{i} > x\right) \sim \sum_{i=1}^{n} \mathbb{P}\left(\theta_{i} X > x\right)$$
$$\sim \overline{F}(x) \sum_{i=1}^{n} \mathbb{E}[\theta_{i}^{\alpha}].$$

Holds even uniformly for n = 1, 2, ...; see Wang (2005).

Example: Stop-loss premium and Value-at-Risk [1]

Let $\widetilde{S}_n = \sum_{i=1}^n \theta_i X_i$. Then

• Stop-loss premium:

$$\mathbb{E}[(\widetilde{S}_n - d)_+] \approx \sum_{i=1}^n \mathbb{E}[(\theta_i X - d)_+].$$

• VaR:

$$\inf\{s: F_{\widetilde{S}_n}(s) \ge p\} \approx$$
$$\inf\left\{s: \sum_{i=1}^n \overline{F}_{\theta_i X}(s) \le 1 - p\right\}.$$

Example: Stop-loss premium and Value-at-Risk [2]

Furthermore, let $F_X \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$. Then

• Stop-loss premium:

$$\mathbb{E}[(\widetilde{S}_n - d)_+] \approx \mathbb{E}[(X - d)_+] \sum_{i=1}^n \mathbb{E}[\theta_i^{\alpha}].$$

i

$$\inf\{s: F_{\widetilde{S}_n}(s) \ge p\} \approx \\ \inf\{s: \overline{F}_X(s) \sum_{i=1}^n \mathbb{E}[\theta_i^{\alpha}] \le 1-p\}.$$

Example: Stop-loss premium and Value-at-Risk [3]

•
$$\theta_n = Y_1 \cdots Y_n$$
, i.i.d.: $\mathbb{E}[\theta_n^{\alpha}] = \mathbb{E}[Y^{\alpha}]^n$.

- $(\theta_1, \ldots, \theta_n) =_d LE_n(\mu_n, \Sigma_n, \phi)$: $\mathbb{E}[\theta_n^{\alpha}]$ is explicit; see e.g., Fang, Kotz & Ng (1990) and Owen & Rabinovitch (1983).
- $(\theta_1, \ldots, \theta_n) =_d LNVMM_n(\mu_n, \beta_n, \Sigma_n, G)$: $\mathbb{E}[\theta_n^{\alpha}]$ is explicit; see e.g., Barndorff-Nielsen (1997).

A numerical illustration

d	"Real"	Appr.	Diff.	Rdiff.	p	"Real"	Appr.	Diff.	Rdiff.
30	1.75	1.56	0.19	11%	0.975	36	30	6	17%
40	1.48	1.35	0.13	9%	0.99	63	57	6	10%
60	1.18	1.11	0.07	6%	0.995	96	90	6	6%
80	1.01	0.96	0.05	5%	0.999	274	265	9	3%
100	0.90	0.86	0.04	4%					
150	0.72	0.70	0.02	3%					
200	0.62	0.61	0.01	2%					
250	0.56	0.55	0.01	2%					
300	0.51	0.50	0.01	2%					

Notes: "Real" versus approximate values of stop-loss premiums and quantiles for Pareto losses and i.i.d. lognormal stochastic discount factors. Fixed parameter values: n = 5, $\alpha = 1.5$, $\mu = -0.04$, $\sigma = 0.10$ and 5,000,000 simulations.

Analytic approximations!

References [1]

Barndorff-Nielsen, Ole E. (1997). "Normal inverse Gaussian distributions and stochastic volatility modelling," *Scandinavian Journal of Statistics* 24, 1-13.

Embrechts, Paul, Claudia Klüppelberg & Thomas Mikosch (1997). *Modelling Extremal Events for Insurance and Finance*, Berlin: Springer.

Fang, Kai T., Samuel Kotz & Kai W. Ng (1990). *Symmetric Multivariate and Related Distributions*, London: Chapman and Hall.

Konstantinides, Dimitrios G. & Thomas Mikosch (2005). "Large deviations and ruin probabilities for solutions to stochastic recurrence equations with heavy-tailed innovations," forthcoming *Annals of Probability*.

References [2]

Owen, Joel & Ramon Rabinovitch (1983). "On the class of elliptical distributions and their applications to the theory of portfolio choice," *Journal of Finance* 38, 745-752.

Rogers, L.C.G. & Z. Shi (1995). "The value of an Asian option," *Journal* of Applied Probability 32, 1077-1088.

Wang, Dingcheng (2005). "Tail probabilities of randomly weighted sums of random variables with dominated variation," forthcoming *Stochastic Models*.