

Web appendix to

“Exchange market pressure in interest rate rules”

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A DSGE model

A.1 Households

The world is populated with a continuum of households, where the population in the home country H lies in the segment $[0, n)$, while that of the rest of the world F is in $[n, 1]$. Domestic households maximize expected lifetime utility

$$\max \mathbb{E}_t \sum_{k=0}^{\infty} \beta^k \left(\frac{C_{t+k}^{1-\sigma}}{1-\sigma} - \frac{L_{t+k}^{1+\gamma}}{1+\gamma} \right), \quad (46)$$

subject to a budget constraint (specified later), by choosing a path $\{C_{t+k}, L_{t+k}\}_{k=0, \dots, \infty}$, where C_{t+k} is household consumption and L_{t+k} is labor supply at time $t+k$.

Consider period t .²¹ Consumption enters the domestic household’s utility as an index C_t , which is the CES aggregate of the indices of domestic consumption of home and foreign (imported) goods, C_{Ht} and C_{Ft} , respectively:

$$C_t = \left(\alpha^{\frac{1}{\eta}} C_{Ht}^{\frac{\eta-1}{\eta}} + (1-\alpha)^{\frac{1}{\eta}} C_{Ft}^{\frac{\eta-1}{\eta}} \right)^{\frac{\eta}{\eta-1}}. \quad (47)$$

The parameter α , determining the preference for home-produced goods, increases with the size of the home country, n , and with home bias ν . We model $1-\alpha = (1-n)(1-\nu)$. Hence, $\nu > 0$ means that domestic households consume fewer foreign-produced goods than the size of the foreign country implies, reflecting home bias.²²

The index of domestic consumption of home goods, C_{Ht} , is the CES aggregate of the consumption of all varieties produced in country H . These are varieties $j \in [0, n)$. The index of domestic consumption of foreign goods, C_{Ft} , is a similar CES aggregate, but concerning all varieties produced in F , which are $j \in [n, 1]$. Domestic consumption

²¹Results for $t+k \geq t$ follow by substituting t by $t+k$, while keeping expectations conditional on t .

²²Because foreign households have identical preferences, their consumption index C_t^* equals the right-hand side of (47) with α substituted by α^* , C_{Ht} by C_{Ht}^* , and C_{Ft} by C_{Ft}^* . Moreover, $\alpha^* = n(1-\nu)$.

of variety j is denoted by $C_t(j)$. In formula,

$$\begin{cases} C_{Ht} = \left[\left(\frac{1}{n} \right)^{\frac{1}{\theta}} \int_0^n C_t(j)^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \\ C_{Ft} = \left[\left(\frac{1}{1-n} \right)^{\frac{1}{\theta}} \int_n^1 C_t(j)^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}}. \end{cases} \quad (48)$$

So, θ concerns the substitutability between varieties produced within a country, whereas η in (47) is about the substitution between home and foreign goods.

As usual, utility maximization requires that within period t households maximize C_t for a given expenditure on home and foreign indices and they maximize C_{Ht} (C_{Ft}) for a given level of expenditure on home (foreign) varieties. Let $P_t(j)$ denote the price of variety j in domestic currency. The resulting demand function for each variety is

$$C_t(j) = \begin{cases} \alpha \left(\frac{P_t(j)}{P_{Ht}} \right)^{-\theta} \left(\frac{P_{Ht}}{P_t} \right)^{-\eta} C_t, & \text{for home varieties } j \in [0, n] \\ (1 - \alpha) \left(\frac{P_t(j)}{P_{Ft}} \right)^{-\theta} \left(\frac{P_{Ft}}{P_t} \right)^{-\eta} C_t, & \text{for foreign varieties } j \in [n, 1], \end{cases} \quad (49)$$

where

$$\begin{cases} P_{Ht} = \left[\frac{1}{n} \int_0^n P_t(j)^{1-\theta} dj \right]^{\frac{1}{1-\theta}} \\ P_{Ft} = \left[\frac{1}{1-n} \int_n^1 P_t(j)^{1-\theta} dj \right]^{\frac{1}{1-\theta}} \end{cases} \quad (50)$$

are the home producer price index and the foreign producer price index expressed in domestic currency, respectively, and

$$P_t = \left(\alpha P_{Ht}^{1-\eta} + (1 - \alpha) P_{Ft}^{1-\eta} \right)^{\frac{1}{1-\eta}} \quad (51)$$

is the consumer price index in the home country. This implies that total consumption expenditure by domestic households is $P_t C_t$.²³

We can now specify the period budget constraint

$$P_t C_t + \mathbb{E}_t \{ \Lambda_{t,t+1} S_{t+1} B_{t+1} \} \leq W_t L_t + S_t B_t + \Pi_t - T_t, \quad (52)$$

where we rule out Ponzi schemes. Here B_t is the value in foreign currency of a portfolio of a full set of state-contingent assets held at the beginning of period t , reflecting our

²³Similar expressions hold for the foreign country, for both demand and prices. Foreign demand follows from (49) by substituting α, C , and the four P symbols by α^*, C^* , and P^* , respectively. The home producer price index in foreign currency P_{Ht}^* and the foreign producer price index (in foreign currency) P_{Ft}^* follow from the right-hand sides of (50) by substituting $P_t(j)$ by $P_t^*(j)$. The foreign consumer price index P_t^* (in foreign currency) equals the right-hand side of (51) with α substituted by α^* , P_{Ht} by P_{Ht}^* , and P_{Ft} by P_{Ft}^* .

complete markets assumption, $S_t = \exp(s_t)$ is the nominal exchange rate in level form, $\Lambda_{t,t+1}$ is the stochastic discount factor making $\mathbb{E}_t \{\Lambda_{t,t+1} S_{t+1} B_{t+1}\}$ the home-currency value at time t of the portfolio that yields a payoff in $t+1$, W_t is the nominal wage, Π_t is nominal firm profits transferred to households, and T_t is lump-sum taxes.

As usual, the first-order conditions consist of the optimality condition regarding the intratemporal consumption-leisure trade off

$$C_t^\sigma L_t^\gamma = \frac{W_t}{P_t} \quad (53)$$

and the intertemporal optimality relation linking the stochastic discount factor to the intertemporal marginal rate of substitution in Euler equation

$$\Lambda_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}}, \quad (54)$$

for all possible states of nature at times t and $t+1$. Note that $\mathbb{E}_t \Lambda_{t,t+1}$ is the value of a portfolio that yields one unit of the domestic currency in $t+1$ (mimicking a riskless domestic bond), so that the interest rate is $i_t = -\log(\mathbb{E}_t \{\Lambda_{t,t+1}\})$. Given i_t , prices, and the budget constraint, the (expectational) Euler equation determines C_t .

A.2 Firms

Firms use labor supplied by the households and a linear technology. Hence, output is

$$Y_t(j) = A_t L_t(j), \quad (55)$$

where A_t is exogenous labor productivity, which is common across firms (within a country). Because of a labor subsidy τ , financed by taxes T_t , marginal cost is

$$MC_t = (1 - \tau) W_t / A_t, \quad (56)$$

which is independent of output and thus common across firms. The firm sells its good in a monopolistically competitive market with free international trade. Profits are

$$\Pi_t(j) = (P_t(j) - MC_t) Y_t(j). \quad (57)$$

The firm sets the price in a sticky fashion a la Calvo (1983). That is, each date with probability ω the firm is not allowed to change its price. When the firm is allowed to set a new price $P_t^{opt}(j)$, it will do so optimally, that is, by maximizing the current market value of the profits resulting while that price remains in place. Suppose the new

price holds until $t + k \geq t$. Let $Y_{t+k|t}(j)$ denote total demand $C_{t+k}(j) + \frac{1-n}{n}C_{t+k}^*(j)$ evaluated at $P_t^{opt}(j)$. The firm's objective function is therefore

$$\max \sum_{k=0}^{\infty} \omega^k \mathbb{E}_t \left\{ \Lambda_{t,t+k} \left(P_t^{opt}(j) - MC_{t+k} \right) Y_{t+k|t}(j) \right\}. \quad (58)$$

To derive the first-order condition, first note that (49) and its foreign counterpart imply that $\partial Y_{t+k|t}(j) / \partial P_t^{opt}(j) = -\theta Y_{t+k|t}(j) / P_t^{opt}(j)$. Moreover, other home firms face the same optimization problem, so that all domestic firms will choose the same new price $P_{Ht}^{opt} = P_t^{opt}(j)$. The price can be solved from the first-order condition

$$\sum_{k=0}^{\infty} \omega^k \mathbb{E}_t \left\{ \Lambda_{t,t+k} Y_{t+k|t} \left[P_{Ht}^{opt} - \frac{\theta}{\theta-1} MC_{t+k} \right] \right\} = 0. \quad (59)$$

A.3 Equilibrium

World equilibrium requires that labor, asset, and goods markets are in equilibrium.

A.3.1 Labor market

Labor market equilibrium at home and abroad requires

$$\begin{cases} L_t &= \frac{1}{n} \int_0^n L_t(j) dj \\ L_t^* &= \frac{1}{1-n} \int_n^1 L_t^*(j) dj. \end{cases} \quad (60)$$

A.3.2 Asset market

As for the home country, market completeness implies there is also a unique stochastic discount factor for foreign-currency payoffs, which is $\Lambda_{t,t+1}^* = \beta (C_{t+1}^*/C_t^*)^{-\sigma} P_t^*/P_{t+1}^*$ for all possible states of nature at times t and $t + 1$. Given free international trade in assets, arbitrage yields the asset market equilibrium relation

$$\Lambda_{t,t+1} = \Lambda_{t,t+1}^* \frac{S_t}{S_{t+1}}, \quad (61)$$

which is a stochastic version of uncovered interest parity.

Substituting the expressions for $\Lambda_{t,t+1}$ and $\Lambda_{t,t+1}^*$ shows the model has the familiar perfect risk sharing relation between home and foreign households

$$C_t^\sigma = C_t^{*\sigma} Q_t, \quad (62)$$

assuming symmetric initial conditions, where $Q_t = S_t P_t^*/P_t$ is the real exchange rate.

A.3.3 Goods market

Goods market equilibrium consists of two parts. First, frictionless trade results in the law of one price. So, for each variety $j \in [0, 1]$ the price set by the producer in its currency implies that the price in the other currency fulfills

$$P_t(j) = S_t P_t^*(j). \quad (63)$$

For the producer price indices this yields $P_{Ht} = P_{Ht}^* S_t$ and $P_{Ft} = P_{Ft}^* S_t$. Still, home bias implies $\alpha > \alpha^*$, so that in general for the consumer price index $P_t \neq P_t^* S_t$, meaning a deviation from purchasing power parity.

The second part of goods market equilibrium is the markets for all varieties clear:

$$\begin{cases} Y_t(j) &= C_t(j) + \frac{1-n}{n} C_t^*(j), \text{ for home varieties} \\ Y_t^*(j) &= \frac{n}{1-n} C_t(j) + C_t^*(j), \text{ for foreign varieties.} \end{cases} \quad (64)$$

For the home-varieties line, substitute the top demand function of (49) for $C_t(j)$ and its foreign counterpart (as explained in footnote 23) for $C_t^*(j)$. For the foreign-varieties line, we do the same, but now using the bottom demand function of (49). This yields

$$\begin{cases} Y_t(j) &= \left(\frac{P_t(j)}{P_{Ht}}\right)^{-\theta} \left[\alpha \left(\frac{P_{Ht}}{P_t}\right)^{-\eta} C_t + \frac{1-n}{n} \alpha^* \left(\frac{P_{Ht}/S_t}{P_t^*}\right)^{-\eta} C_t^* \right], \text{ home v.} \\ Y_t^*(j) &= \left(\frac{P_t^*(j)}{P_{Ft}^*}\right)^{-\theta} \left[\frac{n}{1-n} (1-\alpha) \left(\frac{P_{Ft}^* S_t}{P_t}\right)^{-\eta} C_t + (1-\alpha^*) \left(\frac{P_{Ft}^*}{P_t^*}\right)^{-\eta} C_t^* \right], \text{ foreign v.} \end{cases} \quad (65)$$

Substituting these into the definitions of aggregate output

$$\begin{cases} Y_t &= \left[\frac{1}{n} \int_0^n Y_t(j)^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \\ Y_t^* &= \left[\frac{1}{1-n} \int_n^1 Y_t^*(j)^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \end{cases} \quad (66)$$

gives

$$\begin{cases} Y_t &= \alpha \left(\frac{P_{Ht}}{P_t}\right)^{-\eta} C_t + \frac{1-n}{n} \alpha^* \left(\frac{P_{Ht}/S_t}{P_t^*}\right)^{-\eta} C_t^* \\ Y_t^* &= \frac{n}{1-n} (1-\alpha) \left(\frac{P_{Ft}^* S_t}{P_t}\right)^{-\eta} C_t + (1-\alpha^*) \left(\frac{P_{Ft}^*}{P_t^*}\right)^{-\eta} C_t^* \end{cases}. \quad (67)$$

A.4 Taking the limit $n \rightarrow 0$ to obtain the small economy

To mimic the small open economy we take the limit $n \rightarrow 0$. This implies $\alpha \rightarrow \nu$ and $\alpha^* \rightarrow 0$. The limiting CPIs resulting from (51) become

$$\begin{cases} P_t = \left(\nu P_{Ht}^{1-\eta} + (1-\nu) P_{Ft}^{1-\eta} \right)^{\frac{1}{1-\eta}} \\ P_t^* = P_{Ft}^*, \end{cases} \quad (68)$$

and the limiting values of aggregate output in (67) are

$$\begin{cases} Y_t = \nu \left(\frac{P_{Ht}}{P_t} \right)^{-\eta} C_t + (1-\nu) \left(\frac{P_{Ht}/S_t}{P_t^*} \right)^{-\eta} C_t^* \\ Y_t^* = C_t^*. \end{cases} \quad (69)$$

A.5 Steady state

Here we compute the symmetric zero-inflation and zero-depreciation efficient steady state of the model. All variables refer to the values in that steady state. Similar values apply to the foreign country, unless explicitly stated otherwise.

The constancy of P_H implies that firms choose $P_H^{opt} = P_H$. From (59) we obtain that MC is constant. Given that all shocks are set to zero, productivity is constant over time, denoted by A . Then (56) gives that the wage is constant W . Note the labor subsidy τ given in Table 1 implies $P_H = W/A$ and thus renders the steady state efficient, and real marginal cost $MC/P_H = (\theta - 1)/\theta$. Similarly, $P_F^* = W^*/A^*$. The constancy of S then gives $P_F = SP_F^*$. For simplicity, we assume $P_H = P_F$. So, $P = P_H$. As $P^* = P_F^*$, the real exchange rate $Q = 1$, so that PPP holds in the steady state.

Because all firms j charge the same price, (65) implies that output per firm is the same across varieties. First, consider the foreign country. Combining (65), (67) with the foreign version of (55), (60), (69), and the foreign version of (53), where $W^*/P^* = A^*$, implies that consumption is constant $C^* = A^{*(1+\gamma)/(\sigma+\gamma)}$. Similarly, combining the home equivalents of the formulas in the previous sentence and using the constancy of C^* shows that also home consumption is constant, where C is the unique solution from $C^\sigma ([\nu C + (1-\nu) C^*]/A)^\gamma = A$. Assuming $A = A^*$ and using the value of C^* yields as unique solution $C = C^*$. From (69) we obtain $Y = C$, and (65), (67), (55), and (60) then yield $L = Y/A$. Finally, (54) gives $\Lambda = \beta$.

B Derivations of equations (28)-(39)

Equations (28), (33), (34), (36), and (38) follow directly from (53), (62), (63), (69), and (68), respectively. This appendix derives the remaining equations.

- (29)

Start from (54). By definition, $\mathbb{E}_t \{\Lambda_{t,t+1}\} = \exp(-i_t)$ and $\beta = \exp(-\delta)$. Substitution

into (54) gives

$$1 = \mathbb{E}_t \{ \exp (i_t - \delta - \sigma (c_{t+1} - c_t) - \pi_{t+1}) \}, \quad (70)$$

so that log-linearization yields (29):

$$1 = \mathbb{E}_t \{ 1 + i_t - \delta - \sigma (c_{t+1} - c_t) - \pi_{t+1} \}. \quad (71)$$

• (30)

This results from the log of real marginal cost (56) and employment subsidy $\tau = 1/\theta$.

• (31)

Producer prices are set by firms based on the Calvo structure, so that

$$P_{Ht} = \left[\omega P_{H,t-1}^{1-\theta} + (1-\omega) P_{Ht}^{opt1-\theta} \right]^{\frac{1}{1-\theta}}. \quad (72)$$

Log-linearization yields

$$\pi_{Ht} = (1-\omega) \left(p_{Ht}^{opt} - p_{H,t-1} \right). \quad (73)$$

The optimal price p_{Ht}^{opt} is the solution from the firm's first-order condition. First, rewrite first-order condition (59) as

$$\sum_{k=0}^{\infty} \omega^k \mathbb{E}_t \left\{ \Lambda_{t,t+k} Y_{t+k|t} \left[\frac{P_{Ht}^{opt}}{P_{H,t-1}} - \frac{\theta}{\theta-1} \frac{P_{H,t+k}}{P_{H,t-1}} MC_{H,t+k} \right] \right\} = 0, \quad (74)$$

where we have scaled some variables to obtain ratios that have a well-defined value in the steady state, and $MC_{H,t+k} = MC_{t+k}/P_{H,t+k}$ is the real marginal cost. Using that $\Lambda_{t,t+k} = \beta^k$ in the steady state, log-linearization yields

$$p_{Ht}^{opt} - p_{H,t-1} = (1-\omega\beta) \sum_{k=0}^{\infty} (\omega\beta)^k \mathbb{E}_t \{ p_{H,t+k} - p_{H,t-1} + \widehat{mc}_{H,t+k} \}, \quad (75)$$

where $\widehat{mc}_{H,t+k}$ denotes the deviation of log real marginal cost $mc_{t+k} - p_{H,t+k}$ from its steady state $\log(\frac{\theta-1}{\theta})$. Writing this equation recursively yields

$$p_{Ht}^{opt} - p_{H,t-1} = \omega\beta \mathbb{E}_t \{ p_{H,t+1}^{opt} - p_{Ht} \} + \pi_{Ht} + (1-\omega\beta) \widehat{mc}_{Ht}. \quad (76)$$

Substitution into (73) gives (31), where κ_{mc} is defined in Table 1.

• (32)

Start from the domestic labor market equilibrium formula in (60) and substitute (55). Also substitute $Y_t(j) = \left(\frac{P_t(j)}{P_{Ht}}\right)^{-\theta} Y_t$, which is implied by (65) and (67), and use that $A_t(j) = A_t$. That gives

$$L_t = \frac{1}{n} \int_0^n \left(\frac{P_t(j)}{P_{Ht}}\right)^{-\theta} dj \frac{Y_t}{A_t}. \quad (77)$$

Finally, we take the logarithm and use Galí (2008, p.162) to motivate why the integral part is approximately zero.

• (35)

This results from log-linearizing home aggregate output in (69), which yields

$$y_t = \nu [c_t - \eta (p_{Ht} - p_t)] + (1 - \nu) [c_t^* - \eta (p_{Ht} - s_t - p_t^*)], \quad (78)$$

and then substituting (37) and the law of one price (34).

• (37)

For $\eta \neq 1$, this follows by log-linearizing the top equation in (68). For $\eta = 1$, (68) becomes the Cobb-Douglas combination $P_t = P_{Ht}^\nu P_{Ft}^{1-\nu}$, which also yields (37).

• (39)

The exchange rate s_t clears the asset market, so we start from risk sharing (33). This is the core equation. After substitution of (37) and (38), it is

$$\sigma (c_t - c_t^*) = s_t + p_{Ft}^* - [\nu p_{Ht} + (1 - \nu) p_{Ft}]. \quad (79)$$

This is not yet an s -function in the (i_t, E_t) -form defined in Section 2.2.1, where the i_t argument captures the interest rate impact on the exchange rate via all contemporaneous channels, and the vector E_t accounts for everything else. Therefore, we now substitute out contemporaneous variables affected by i_t . We do so in a streamlined manner, where first-order conditions and equilibrium relations are used only once, and they substitute out the choice and equilibrating variables based on the underlying economic mechanisms.

We first focus on p_{Ht} . Calvo pricing (31) and marginal cost (30) imply

$$p_{Ht} = p_{H,t-1} + \beta \mathbb{E}_t \{\pi_{H,t+1}\} + \kappa_{mc} (w_t - a_t - p_{Ht}), \quad (80)$$

reflecting that p_{Ht} is driven by wage w_t . The latter is such that labor demand equals supply, reflected by labor market equilibrium (32). Households' labor supply ℓ_t satisfies

(28). Combining them and using (37) gives the equilibrium wage

$$w_t = \nu p_{Ht} + (1 - \nu) p_{Ft} + \gamma (y_t - a_t) + \sigma c_t, \quad (81)$$

so that w_t depends on output y_t . The latter equilibrates the goods market. Substituting goods market equilibrium (35) for y_t in the equilibrium wage yields product wage

$$w_t - p_{Ht} = \varpi_{tot} (p_{Ft} - p_{Ht}) + \varpi_c c_t + \gamma [(1 - \nu) c_t^* - a_t], \quad (82)$$

where ϖ_c (ϖ_{tot}) is the full impact of c_t (tot_t) on the product wage for given terms of trade (consumption), as defined in Table 1. Substitution into (80) yields

$$p_{Ht} = \frac{1}{1 + \kappa_{mc} \varpi_{tot}} \left[\begin{array}{l} p_{H,t-1} + \beta \mathbb{E}_t \{ \pi_{H,t+1} \} \\ + \kappa_{mc} (\varpi_{tot} p_{Ft} + \varpi_c c_t + \gamma (1 - \nu) c_t^* - (\gamma + 1) a_t) \end{array} \right]. \quad (83)$$

Substituting this for p_{Ht} and (34) for p_{Ft} in core equation (79) gives

$$\begin{aligned} \sigma (c_t - c_t^*) &= s_t + p_{Ft}^* - \nu \frac{1}{1 + \kappa_{mc} \varpi_{tot}} \left[\begin{array}{l} p_{H,t-1} + \beta \mathbb{E}_t \{ \pi_{H,t+1} \} \\ + \kappa_{mc} (\varpi_{tot} (p_{Ft}^* + s_t) + \varpi_c c_t) \\ + \kappa_{mc} (\gamma (1 - \nu) c_t^* - (\gamma + 1) a_t) \end{array} \right] \\ &\quad - (1 - \nu) (p_{Ft}^* + s_t). \end{aligned} \quad (84)$$

Next, focus on the foreign price p_{Ft}^* . It follows similarly as p_{Ht} , using the foreign equivalents of (31), (30), (32), (28), and using (38) and (36) instead of (37) and (35). This gives

$$p_{Ft}^* = p_{F,t-1}^* + \beta \mathbb{E}_t \{ \pi_{F,t+1}^* \} + \kappa_{mc} ((\sigma + \gamma) c_t^* - (\gamma + 1) a_t^*). \quad (85)$$

Substituting (85) for p_{Ft}^* in (84) and then using the lag of (34) for $p_{F,t-1}^*$ and Euler equation (29) and its foreign equivalent to remove $c_t - c_t^*$ yields (39).

To understand that this is in (i_t, E_t) -form, realize that all predetermined, exogenous, and foreign variables are unaffected by i_t , so they have to be put in E_t . The recursive nature of (29) implies that $\mathbb{E}_t \{ c_{t+1} \}$ is determined by expectations of future variables, so there is no contemporaneous effect of i_t , making $\mathbb{E}_t \{ c_{t+1} \}$ part of E_t . Similarly, (31) implies that $\mathbb{E}_t \{ \pi_{H,t+1} \}$ is part of E_t . For $\mathbb{E}_t \{ \pi_{t+1} \}$ one should realize that households base their consumption decision on $\mathbb{E}_t \{ \pi_{t+1} \}$ as a whole, not on just the p_t part within it. Hence, i_t can only affect c_t via $\mathbb{E}_t \{ \pi_{t+1} \}$ if the latter as a whole changes, so that $\mathbb{E}_t \{ \pi_{t+1} \}$ does not contain a contemporaneous channel and is thus part of E_t .