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Bartlett Correction in the Stable AR(1) Model with Intercept and Trend

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Abstract

The Bartlett correction is derived for testing hypotheses about the autoregressive parameter ρ in the stable: (i) AR(1) model; (ii) AR(1) model with intercept; (iii) AR(1) model with intercept and linear trend. The correction is found explicitly as a function of ρ. In the models with deterministic terms, the correction factor is asymmetric in ρ. Furthermore, the Bartlett correction is monotonically increasing in ρ and tends to infinity when ρ approaches the stability boundary of $+1$. Simulation results indicate that the Bartlett corrections are useful in controlling the size of the LR statistic in small samples, although these corrections are not the ultimate panacea.

Keywords: Autoregressive models, Bartlett correction, small-sample properties, likelihood ratio statistic
JEL classification: C13; C22.

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1 INTRODUCTION

Several decades ago, Lawley (1956) has shown in the i.i.d. setup that the finite-sample distribution of a Bartlett (1937) corrected likelihood ratio (LR) test statistic is closer to the χ^2-distribution than the original LR statistic; see Cribari-Neto and Cordeiro (1996) for an econometric oriented review. Recently, Bartlett-*type* corrections in *un*stable autoregressive models have attracted much attention, see inter alias Bravo (1999), Nielsen (1997), Larsson (1998) and Johansen (2004), although Jensen and Wood (1997) have shown that the usual conditions for a Bartlett correction are not fulfilled in the unstable first-order autoregressive –AR(1)– model.

In this paper, we analyze the stable AR(1) model, which without deterministic terms was also considered by Taniguchi (1988, 1991), Omtzigt (2003) and Lagos and Morettin (2004). However, in contrast to earlier work, we also consider the AR(1) model with intercept (and linear trend). Anticipating the results, we can say that the correction factor in models with deterministic components is considerably different from the factor obtained in the pure AR(1) model.

Although a number of exact inference techniques are available for the AR(1) model, see inter alias Andrews (1993) and Kiviet and Dufour (1997), these methods are based on numerical (Monte Carlo) procedures that only give limited analytical insight into the structure of the finite-sample problems. Furthermore, analytical techniques like Edgeworth expansions, see e.g. Phillips (1977), or saddlepoint approximations often lead to very complicated formulas, which are difficult to interpret.

Suppose that the likelihood function, whose logarithm will be denoted by L, depends upon $p+1$ population parameters $\theta_1, ..., \theta_p, \theta_{p+1}$. In our case, $\theta_1, ..., \theta_p$ will denote the nuisance parameters like the error variance, intercept and trend coefficient, while θ_{p+1} is the parameter of interest namely the first-order autocorrelation coefficient ρ. For testing the null hypothesis $H_0 : \rho = \rho_0$, the LR statistic can be written as $2(L^{(p+1)} - L^{(p)})$, where $L^{(k)}$ denotes the result of maximizing L with respect to $\theta_1, ..., \theta_k$ and substituting true values under the null for the remaining parameters. Under some continuity assumptions on the likelihood and its derivatives, together with the assumption that the second-order derivatives of L with respect to the parameters are of order T (the sample size), Lawley (1956) has shown that under H_0 the expectation of the LR statistic may be written as

$$\mathbb{E}[LR] = 1 + \xi_{p+1} - \xi_p + O(T^{-2}),$$

where

$$\xi_k = \sum_{r,s,t,u=1}^{k} \lambda^{rs}\lambda^{tu}\{\tfrac{1}{4}\lambda_{rstu} - (\lambda_{rst})_u + (\lambda_{rt})_{su}\} - \sum_{r,s,t,u,v,w=1}^{k} \lambda^{rs}\lambda^{tu}\lambda^{vw}\{\tfrac{1}{6}\lambda_{rtv}\lambda_{suw} + \tfrac{1}{4}\lambda_{rtu}\lambda_{svw} - \lambda_{rtv}(\lambda_{sw})_u - \lambda_{rtu}(\lambda_{sw})_v + (\lambda_{rt})_v(\lambda_{sw})_u + (\lambda_{rt})_u(\lambda_{sw})_v\}, \quad (1)$$

with

$$\begin{aligned} L_{rs} &= \partial^2 L/\partial\theta_r\partial\theta_s, & L_{rst} &= \partial^3 L/\partial\theta_r\partial\theta_s\partial\theta_t, \quad \text{etc.,} \\ \lambda_{rs} &= \mathbb{E}[L_{rs}], & \lambda_{rst} &= \mathbb{E}[L_{rst}], \quad \text{etc.,} \\ (\lambda_{rs})_t &= \partial\lambda_{rs}/\partial\theta_t, & (\lambda_{rst})_u &= \partial\lambda_{rst}/\partial\theta_u, \quad \text{etc.,} \\ (\lambda_{rs})_{tu} &= \partial^2\lambda_{rs}/\partial\theta_t\partial\theta_u, \end{aligned}$$

and $[\lambda^{rs}]$ is the inverse matrix of $[\lambda_{rs}]$.

In the class of stationary Gaussian autoregressive moving average models, Taniguchi (1988) has shown that the LR test is the only Bartlett correctable test among the LR, Wald, modified Wald and Lagrange multiplier test , i.e.

$$\mathbb{P}[LR^* \le x] = P[\chi_1^2 \le x] + o(T^{-1}), \quad (2)$$

where $LR^* = LR/(1 + \xi_{p+1} - \xi_p)$ denotes the Bartlett corrected LR statistic. Since all Bartlett corrections are calculated according to the general formula shown in (1) and the dynamics in our models are of the autoregressive type, the higher-order result stated in (2) should also apply to our Bartlett corrected LR statistics.

The paper is organized as follows. In sections 2-5, the Bartlett corrections are derived in the AR(1) models with and without intercept and linear trend. Section 6 presents some simulation results to shed some light on the small-sample properties of the Bartlett-corrected tests, while Section 7 concludes.

A word on notation. Throughout this paper, the symbol $\stackrel{1}{=}$ will indicate that we have kept terms of order T^{-1}, i.e. if the stochastic expansion is of the form $V = V_0 + T^{-1}V_1 + T^{-2}V_2 + ...$, then $V \stackrel{1}{=} V_0 + T^{-1}V_1$, where $V_i \in O_p(1)$ are random variables. Furthermore, we use $\sum$ to indicate summation over $t = 1, ..., T$.

2 THE AR(1) MODEL

First, consider the asymptotic stationary, mean zero, Gaussian AR(1) model

$$y_t = \rho y_{t-1} + \varepsilon_t, \qquad t = 1, ..., T, \qquad |\rho| < 1. \tag{3}$$

Inference is made conditional on the starting value y_0, so that ρ can be efficiently estimated by OLS. The LR statistic for testing $H_0 : \rho = \rho_0$ against $H_1 : \rho \neq \rho_0$ in this model is equal to

$$LR = -T \log(1 - r_{y\varepsilon}^2), \tag{4}$$

where

$$r_{y\varepsilon} \equiv \frac{\sum y_{t-1}\varepsilon_t}{\sqrt{\sum y_{t-1}^2 \sum \varepsilon_t^2}}$$

denotes the sample correlation coefficient between y_{t-1} and ε_t assuming zero means; see for instance formula (4) of Nielsen (1997). Note that (y_t/σ) is an AR(1) process with (ε_t/σ) as innovations and starting value (y_0/σ). Therefore, $r_{y\varepsilon}$ (and hence the LR statistic and its Bartlett correction) is scale invariant. The Bartlett correction in the pure AR(1) model is stated in Theorem 1.

Theorem 1 *In the stable Gaussian AR(1) model, given in (3), the expectation of the likelihood ratio test for the null hypothesis $\rho = \rho_0$ has the expansion*

$$\mathbb{E}[LR] \stackrel{1}{=} 1 - \frac{1}{6T} - \frac{1}{3T} = 1 - \frac{1}{2T}. \tag{5}$$

Proof of Theorem 1. We have to determine ξ_1 and ξ_2 in formula (1) for $(\theta_1, \theta_2) = (\sigma^2, \rho)$. Since all the λ's in the pure AR(1) model are a subset of the λ's in the AR(1) model with intercept and trend, the λ's are only shown in the proof of Theorem 3. Since the index of ρ changes in the various models (for this particular model it is 2), the subindex of this parameter is replaced by ρ. The following λ's are non-zero: $\lambda_{11}, \lambda_{\rho\rho}, \lambda_{111}, \lambda_{(1\rho\rho)}, \lambda_{1111}, \lambda_{(11\rho\rho)}$. Here, the subindex $(1\rho\rho)$ means all permutations of $(1\rho\rho)$, so that $\lambda_{(1\rho\rho)} \in \{\lambda_{1\rho\rho}, \lambda_{\rho 1\rho}, \lambda_{\rho\rho 1}\}$. Carrying out the summation shown in (1) for $k = 2$ and $k = 1$ gives $\xi_2 = -1/(6T)$ and $\xi_1 = 1/(3T)$, which completes the proof. □

Somewhat surprisingly, the Bartlett correction turns out to be independent of the AR(1) parameter ρ. Furthermore, the factor is always smaller than 1. Hence, the Bartlett correction predicts that the uncorrected LR statistic tends to underreject a correct null hypothesis, i.e. its rejection probability tends to be lower than the nominal level.

Note that the result in (5) is different from Taniguchi (1988), which does not treat σ^2 as a nuisance parameter. In fact, if σ^2 is assumed to be known, we have $\lambda_{\rho\rho} = -(1-\rho^2)^{-1}T$ and $\lambda_{\rho\rho\rho} = \lambda_{\rho\rho\rho\rho} = 0$, so that formula (1) for $k = 1$ reduces to

$$\xi_1 = \frac{(\lambda_{\rho\rho})_{\rho\rho}}{\{\lambda_{\rho\rho}\}^2} - 2\frac{\{(\lambda_{\rho\rho})_{\rho}\}^2}{\{\lambda_{\rho\rho}\}^3} = -\frac{2}{T}.$$

This leads to a Bartlett factor of $1 - 2/T (= 1 + \xi_1 - \xi_0)$, which is the correction term reported in Taniguchi (1988, p. 504).

3 AR(1) MODEL WITH INTERCEPT

In this section, the AR(1) process is (asymptotically) stationary around a non-zero mean. Consider the AR(1) model with an intercept

$$(y_t - \mu) = \rho(y_{t-1} - \mu) + \varepsilon_t, \qquad \varepsilon_t \sim \text{NID}(0, \sigma^2), \qquad |\rho| < 1. \tag{6}$$

As before, inference is conditional on the starting value y_0. To obtain an expression for the LR statistic under H_0, first, demean the series $\{y_{t-1}\}$ and $\{\varepsilon_t\}$. From Andrews (1993, Appendix A), it follows that $\tilde{y}_{t-1} \equiv y_{t-1} - T^{-1}\sum y_{t-1}$ and $\tilde{\varepsilon}_t \equiv \varepsilon_t - T^{-1}\sum \varepsilon_t$ are invariant with respect to the value of μ. Analogously to (4), the LR statistic is given by

$$LR^c = -T\log(1 - (r^c_{y\varepsilon})^2),$$

where

$$r^c_{y\varepsilon} \equiv \frac{\sum \tilde{y}_{t-1}\tilde{\varepsilon}_t}{\sqrt{\sum \tilde{y}^2_{t-1} \sum \tilde{\varepsilon}^2_t}}$$

denotes the sample correlation coefficient between y_{t-1} and ε_t in deviations from their means. Since $r^c_{y\varepsilon}$ is location and scale invariant, the Bartlett correction shown in Theorem 2 does not depend on (μ, σ^2).

Theorem 2 *In the stable Gaussian AR(1) model with intercept, given in (6), the expectation of the likelihood ratio test for the null hypothesis $\rho = \rho_0$ has the expansion*

$$\mathbb{E}[LR^c] \stackrel{1}{=} 1 + \frac{7 + 5\rho}{3(1-\rho)T} - \frac{11}{6T} = 1 + \frac{1 + 7\rho}{2(1-\rho)T}. \tag{7}$$

Proof of Theorem 2. In this setup, we have $(\theta_1, \theta_2, \theta_3) = (\sigma^2, \mu, \rho)$. The following λ's are non-zero: λ_{11},λ_{22}, $\lambda_{\rho\rho}$, λ_{111}, $\lambda_{(122)}$, $\lambda_{(1\rho\rho)}$, $\lambda_{(22\rho)}$, λ_{1111}, $\lambda_{(1122)}$, $\lambda_{(11\rho\rho)}$, $\lambda_{(122\rho)}$, $\lambda_{(22\rho\rho)}$; see Theorem 3 for their values. As before, a subindex between parentheses means all permutations of that subindex. Carrying out the summation shown in (1) for $k = 3$ and $k = 2$ gives $\xi_3 = (7 + 5\rho)/(3(1 - \rho)T)$ and $\xi_2 = 11/(6T)$. Calculating $1 + \xi_3 - \xi_2$ completes the proof. □

This Bartlett correction agrees with the factor calculated by Van Giersbergen (2004, Theorem 2), where the expectation is determined in a more laborious but direct way. Contrary to the case when there is no intercept, i.e. formula (5), we now see that the factor depends on the AR(1) parameter ρ. For $\rho > -1/3$, the Bartlett correction in the model with intercept is larger than the Bartlett correction in the model without intercept. Furthermore, the factor is increasing in ρ and hence it is asymmetric with respect to the origin; it even has an asymptote for $\rho \uparrow 1$.

4 AR(1) MODEL WITH INTERCEPT AND TREND

Consider the AR(1) model with intercept and linear trend

$$(y_t - \mu - \beta t) = \rho(y_{t-1} - \mu - \beta(t-1)) + \varepsilon_t, \qquad \varepsilon_t \sim \mathrm{NID}(0, \sigma^2), \qquad |\rho| < 1. \tag{8}$$

Inference is again conditional on the starting value y_0. To obtain the LR statistic, first regress y_{t-1} on $(1, t)$ and ε_t on $(1, t)$. From Andrews (1993, Appendix A), it follows that both sets of residuals are invariant with respect to the value of (μ, β) and therefore also the LR statistic. The LR statistic is given by

$$LR^\tau = -T \log(1 - (r^\tau_{y\varepsilon})^2),$$

where $r^\tau_{y\varepsilon}$ denotes the sample correlation coefficient between y_{t-1} and ε_t in deviation from a constant and trend. Theorem 3 gives an expression for the Bartlett factor in the AR(1) model with intercept and trend.

Theorem 3 *In the stable Gaussian AR(1) model with intercept and linear trend, given in (8), the expectation of the likelihood ratio test for the null hypothesis $\rho = \rho_0$ has the expansion*

$$\mathbb{E}[LR^\tau] \overset{1}{=} 1 + \frac{47 + 25\rho}{6(1-\rho)T} - \frac{13}{3T} = 1 + \frac{7 + 17\rho}{2(1-\rho)T}.$$

Proof of Theorem 3. First, divide the linear trend by T, so that the second-order derivatives are of order T as required by Lawley (1956). This can be done without loss of generality since the test statistic is invariant to this transformation. The log-likelihood (conditional on y_0) is given by

$$L \propto -\frac{T}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum(y_t - \mu - \beta(t/T) - \rho(y_{t-1} - \mu - \beta(t-1)/T))^2 . \tag{9}$$

This log-likelihood has the advantage that we may assume that y_t is a zero-mean process (asymptotically), i.e. we may assume $(\mu, \beta) = (0, 0)$ when evaluating the expectations of the derivatives. Although the test statistic is invariant w.r.t. (μ, β), this fact cannot be exploited in the Lawley procedure applied to the log-likelihood associated with the model $y_t = \rho y_{t-1} + \ddot{\mu} + \ddot{\beta}t + \varepsilon_t$, i.e. we then explicitly have to use $y_t \approx \ddot{\mu}/(1-\rho) + \ddot{\beta}/(1-\rho)(t/T) + \sum_{i=0}^{t-1}\rho^i\varepsilon_{t-i}$. So for convenience and to save space, we prefer to use the log-likelihood as formulated in (9). In model (8), we have $(\theta_1, \theta_2, \theta_3, \theta_4) = (\sigma^2, \mu, \beta, \rho)$. All derivatives up to the fourth order were determined using Mathematica 5.0 (see Wolfram (1991)). Due to the invariance and formula (9), we may assume $\mu = \beta = 0$, so that $y_t = \sum_{i=0}^{t-1}\rho^i\varepsilon_{t-i} + \rho^t y_0$. Conditional on y_0, we have

$$\mathbb{E}[\sum y_t] = \sum \rho^t y_0 = \rho\frac{1-\rho^T}{1-\rho}y_0 = O(1).$$

Analogously, we find

$$\mathbb{E}[\sum y_{t-1}] = O(1), \quad \mathbb{E}[\sum(t/T)y_t] = O(1) \text{ and } \mathbb{E}[\sum(t/T)y_{t-1}] = O(1).$$

Furthermore,

$$\begin{aligned}
\mathbb{E}[\sum y_t^2] &= \mathbb{E}[\textstyle\sum_{t=1}^{T}(\sum_{i=0}^{t-1}\rho^i\varepsilon_{t-i} + \rho^t y_0)^2] = \sum_{t=1}^{T}(\sum_{i=0}^{t-1}\rho^{2i}\sigma^2 + \rho^{2t}y_0^2) \\
&= \textstyle\sum_{t=1}^{T}(\frac{1-\rho^{2t}}{1-\rho^2}\sigma^2 + \rho^{2t}y_0^2) \\
&= \sigma^2\frac{(1-\rho^2)T - \rho^2(1-\rho^{2T})}{(1-\rho^2)^2} + \rho^2\frac{1-\rho^{2T}}{1-\rho^2}y_0^2 = \frac{\sigma^2}{(1-\rho^2)}T + O(1),
\end{aligned} \tag{10}$$

so that

$$\mathbb{E}[\sum y_{t-1}^2] = \frac{\sigma^2}{(1-\rho^2)}T + O(1) \quad \text{and} \quad \mathbb{E}[\sum y_t y_{t-1}] = \rho\frac{\sigma^2}{(1-\rho^2)}T + O(1).$$

Since only $O(T)$-terms contribute to the Bartlett correction, the contribution of y_0 is negligible for our purpose. Lastly, we can use the following two approximations

$$\sum(t/T) = \frac{1}{2}T + O(1) \qquad \text{and} \qquad \sum(t/T)^2 = \frac{1}{3T} + O(1).$$

The expectations of the second-order derivatives are given by (after replacing all subindices 4 by ρ and omitting $O(1)$-terms):

$$\begin{aligned} \lambda_{11} &= -\tfrac{1}{2\sigma^4}T, & \lambda_{33} &= -\tfrac{(1+\rho+\rho^2)}{3\sigma^2}T, \\ \lambda_{22} &= -\tfrac{(1-\rho)^2}{\sigma^2}T, & \lambda_{\rho\rho} &= -\tfrac{1}{(1-\rho^2)}T, \\ \lambda_{(23)} &= -\tfrac{(1-\rho^2)}{2\sigma^2}T. \end{aligned}$$

The non-zero expectations of the third-order derivatives are given by (omitting $O(1)$-terms):

$$\begin{aligned} \lambda_{111} &= \tfrac{2}{\sigma^6}T, & \lambda_{(1\rho\rho)} &= \tfrac{1}{(1-\rho^2)\sigma^2}T, \\ \lambda_{(122)} &= \tfrac{(1-\rho)^2}{\sigma^4}T, & \lambda_{(22\rho)} &= \tfrac{2(1-\rho)}{\sigma^2}T, \\ \lambda_{(123)} &= \tfrac{(1-\rho^2)}{2\sigma^4}T, & \lambda_{(23\rho)} &= \tfrac{\rho}{\sigma^2}T, \\ \lambda_{(133)} &= \tfrac{(1+\rho+\rho^2)}{3\sigma^4}T, & \lambda_{(33\rho)} &= -\tfrac{(1+2\rho)}{3\sigma^2}T. \end{aligned}$$

The non-zero expectations of the fourth-order derivatives are given by (omitting $O(1)$-terms):

$$\begin{aligned} \lambda_{1111} &= -\tfrac{9}{\sigma^8}T, & \lambda_{(122\rho)} &= -\tfrac{2(1-\rho)}{\sigma^4}T, \\ \lambda_{(1122)} &= -\tfrac{2(1-\rho)^2}{\sigma^6}T, & \lambda_{(123\rho)} &= -\tfrac{\rho}{\sigma^4}T, \\ \lambda_{(1123)} &= -\tfrac{(1-\rho^2)}{\sigma^6}T, & \lambda_{(22\rho\rho)} &= -\tfrac{2}{\sigma^2}T, \\ \lambda_{(1133)} &= -\tfrac{2(1+\rho+\rho^2)}{3\sigma^6}T, & \lambda_{(33\rho\rho)} &= -\tfrac{2}{3\sigma^2}T. \\ \lambda_{(11\rho\rho)} &= -\tfrac{2}{(1-\rho^2)\sigma^4}T, \end{aligned}$$

Using Mathematica to carry out the summation shown in (1) for $k = 4$ and $k = 3$ gives $\xi_4 = (47 + 25\rho)/(6(1-\rho)T)$ and $\xi_3 = 13/(3T)$. Calculating $1 + \xi_4 - \xi_3$ completes the proof. □

The Bartlett correction in the AR(1) model with intercept and trend has the same functional form as in the model with only an intercept. As before, the factor is increasing in ρ and goes to infinity for $\rho \uparrow 1$. For $\rho > -3/5$, the correction factor in the model with trend is larger than the factor in the model without a trend. Hence, for this range, the finite-sample problems in the model with trend are expected to be more severe than in the model without trend.

5 SOME FURTHER RESULTS

In this section, we shall consider two issues that were brought up by a referee. First, we investigate the effect of using the unconditional likelihood. Secondly, we shall consider some terms that are asymptotically negligible but might be important in finite samples.

Consider the stationary AR(1) model

$$y_t^* = \rho y_{t-1}^* + \varepsilon_t, \qquad |\rho| < 1, \qquad y_0^* \sim N(0, \sigma^2/(1-\rho^2)).$$

The (unconditional) log-likelihood is given by

$$L \propto -\frac{1}{2}\log\left(\frac{\sigma^2}{(1-\rho^2)}\right) - \frac{(1-\rho^2)}{2\sigma^2}(y_0^*)^2 - \frac{T}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum\left(y_t^* - \rho y_{t-1}^*\right)^2,$$

where the first two terms are due to y_0^*. The second-order derivatives L_{11}, $L_{(1\rho)}$ and $L_{\rho\rho}$ are easy to derive. In this model specification, we have the following exact results

$$\begin{aligned}
\mathbb{E}[\sum y_t^*] &= \mathbb{E}[\sum y_{t-1}^*] = 0 \qquad\qquad \text{and} \qquad \mathbb{E}[y_0^*] = 0,\\
\mathbb{E}[\sum (y_t^*)^2] &= \mathbb{E}[\sum (y_{t-1}^*)^2] = \sigma^2/(1-\rho^2)T \qquad \text{and} \qquad \mathbb{E}[\sum y_t^* y_{t-1}^*] = \rho\sigma^2/(1-\rho^2)T,
\end{aligned}$$

and the expectations of the derivatives under the stationary distribution are given by

$$\begin{aligned}
\lambda_{11} &= -\frac{1}{2\sigma^4}T + \frac{1}{2\sigma^4} = -\frac{1}{2\sigma^4}T + O(1), \qquad\qquad \lambda_{(1\rho)} = 0,\\
\lambda_{(\rho\rho)} &= -\frac{1}{(1-\rho^2)}T - \frac{1+\rho^2}{(1-\rho^2)^2} = -\frac{1}{(1-\rho^2)}T + O(1).
\end{aligned}$$

Since only $O(T)$-terms contribute to the Bartlett correction, the expectations of the derivatives are the same as in the previous section. This also holds for the higher-order derivatives and in the models with deterministic components. Therefore, the Bartlett correction based on the unconditional likelihood is the same as the one based on the conditional likelihood.

Next, we consider the pure AR(1) model with $y_0 = 0$. Due to formula (10), we can write

$$\mathbb{E}[\sum y_t^2] = \frac{\sigma^2}{(1-\rho^2)}T - \rho^2\frac{(1-\rho^{2T})}{(1-\rho^2)^2}\sigma^2. \tag{11}$$

Looking at these components from a local-to-unity perspective, i.e. $\rho = 1 - \gamma/T$, we have that the first term, which is $O(T^2)$, is just as large as the second term, which is also $O(T^2)$. Note that this second term was neglected in the analysis so far, which resulted in a constant Bartlett correction. Using Lawley's formula (1), together with the result stated in (11) and

$$\mathbb{E}[\sum y_{t-1}^2] = \frac{\sigma^2}{(1-\rho^2)}T - \frac{(1-\rho^{2T})}{(1-\rho^2)^2}\sigma^2 \qquad \text{and} \qquad \mathbb{E}[\sum y_t y_{t-1}] = \rho\mathbb{E}[\sum y_{t-1}^2],$$

we find the following extended Bartlett correction for the pure AR(1) model

$$\begin{aligned}\mathbb{E}[LR] \;\approx\; & 1+\frac{3}{2T} \\ & +\frac{4\rho^2(1-3\rho^2)(1-\rho^{2T})^2-2(1-\rho^2)(1-\rho^{2T})(3\rho^2(1-\rho^2)-\rho^{2T}(1+7\rho^2))T}{\rho^2(1-\rho^{2T}-(1-\rho^2)T)^3} \\ & -\frac{2(1-\rho^2)^2(2\rho^{4T}+\rho^{2T}(3-\rho^2)-\rho^2(1-\rho^2))T^2-4\rho^{2T}(1-\rho^2)^3T^3}{\rho^2(1-\rho^{2T}-(1-\rho^2)T)^3}. \qquad (12)\end{aligned}$$

Looking at formula (12), the extended Bartlett correction now does depend on ρ. In fact, it remains finite for $\rho \uparrow 1$, since

$$\lim_{\rho\uparrow 1}\mathbb{E}[LR] \approx \frac{1}{18}\left(26+\frac{8}{T-1}-\frac{5}{T}\right). \qquad (13)$$

Unfortunately, formula (13) overestimates the true expectation of the LR statistic in the unit root model; see Nielsen (1997) for a more accurate approximation.

Figure 1 shows various corrections factors including the extended Bartlett correction. Furthermore, it shows the expectation of the LR statistics, i.e. $\mathbb{E}[LR]$, $\mathbb{E}[LR^c]$ and $\mathbb{E}[LR^\tau]$, approximated by simulation and correction factors (based on simulations) that leads to LR statistics having a 5% size. Note that the latter factors are not exact for other significant levels. In the pure AR(1) model, the extended Bartlett correction as shown in (12) increases as $|\rho|$ approaches 1, although it is clearly larger than $\mathbb{E}[LR]$. Furthermore, it seems that the simulated values of $\mathbb{E}[LR]$ are also too large for $|\rho|$ near to 1. In the AR(1) model with intercept (and also trend), there is only a significant different value between the extended and non-extended Bartlett correction for values of ρ close to -1. Furthermore, the simulated values of $\mathbb{E}[LR^c]$ (and $\mathbb{E}[LR^\tau]$) now deviate substantially from the adjustment that leads to a correct size. This shows that the Bartlett approach based on the expectation of the LR statistic is not very adequate in the model containing deterministic terms when ρ is close to 1. In these models, inference based on the restricted likelihood seems promising; see Chen and Deo (2006).

We conclude by noting that the unconditional approach leads to still another extended Bartlett correction for $y_0 \neq 0$, so then the correspondence between the two approaches is lost. Since the unconditional approach uses more information, a comparison between the two is outside the scope of this paper.

Insert Figure 1 about here.

6 MONTE CARLO RESULTS

To assess the quality of the asymptotic expansions, a small simulation study has been carried out. All the simulations were done on a PC using Gauss 6. Observations were generated according to an AR(1) process. All reported results are based on 10^5 replications and two sample sizes were considered: $T \in \{20, 80\}$. The AR(1) parameter was taken as $\rho \in \{-0.99, -0.98, ..., 0.99\}$. Since the LR statistic is invariant with respect to σ (and μ in the model with intercept), we set $\sigma = 1$ (and $\mu = 0$) without loss of generality. In addition, $\beta = 0$ in the model with a trend. The starting value was set to zero, i.e. $y_0 = 0$. The simulations were also carried out for $y_0 \sim N(0, 1/(1-\rho^2))$, which did not change the results significantly. The nominal significance level was taken to be 5%. Results remain qualitatively the same for the 1% and 10% level.

Figure 2 shows the rejection frequencies for the three AR(1) models considered in this paper. In the pure AR(1) model, inference based on the uncorrected LR test and the χ^2-distribution performs reasonably well. Only when $|\rho|$ is very close to 1, the rejection frequencies rise slightly above 6%. Note that the rejection error rate seems to be an even function of ρ. Since the Bartlett correction does not depend on ρ, the curve of rejection frequencies is shifted towards the nominal significance level for $|\rho| < 0.8$. In line with Figure 1, inference based on simulated values of $\mathbb{E}[LR]$ as a correction factor becomes conservative for $|\rho| > 0.9$.

Insert Figure 2 about here.

Next, consider the rejection frequencies when an intercept is added to the AR(1) model (middle two graphs of Figure 2). As expected, the discrepancy between the rejection frequencies and the nominal level is larger than in the model without intercept. Furthermore, the rejection error rate is more pronounced for ρ close to $+1$ than for ρ close to -1. The Bartlett correction seems to work well for $\rho \in (-0.9, 0.7)$ when $T = 20$. For $T = 80$, the frequencies are only 1% different from the nominal level if $\rho \leq 0.9$. Since the Bartlett factor tends to infinity as $\rho \uparrow 1$, the corrected test becomes very conservative for values of ρ close to 1. The use of the simulated values of $\mathbb{E}[LR^c]$ leads to a slight improvement in comparison to the analytical Bartlett factor.

Lastly, we consider the results for the AR(1) model with intercept and trend (lower two graphs of Figure 2). When $T = 20$, the ordinary LR test massively rejects the true null hypothesis for large positive values of ρ (its empirical size is higher than 20% for $\rho > 0.7$). The Bartlett-corrected LR test, however, becomes very conservative for these parameter values. For $T = 80$, the rejection frequencies are within 1 percentage point of the nominal significance level for $\rho \leq 0.8$. Again, adjusting using simulated values of $\mathbb{E}[LR^{\tau}]$ leads to more accurate inference for value of ρ close to 1.

Overall, we conclude that the Bartlett correction works well within a reasonable range of the parameter space. However, the ability to control the size depends on the deterministic terms included in the estimation model.

7 CONCLUSION

In this paper, the Bartlett correction is derived for testing hypotheses about the autoregressive parameter ρ in the stable AR(1) model with and without an intercept and linear trend. In case deterministic components are present, it is found that the correction factor is asymmetric in ρ. Furthermore, the Bartlett correction is monotonically increasing in ρ and tends to infinity when ρ approaches the stability boundary of $+1$. Hence, the Bartlett factor overcorrects for large positive values of ρ. At least theoretically, the (first-order) Bartlett correction does not depend on the treatment of the starting value.

The simulation results indicate that the Bartlett corrections are useful for controlling the size of the LR statistic in the models considered. The empirical size is close to the nominal significance level (only 1 percentage point deviation) for a large part of the parameter space, even though the range of the parameter space critically depends upon the deterministic components in the estimation model. Although useful, these Bartlett corrections are not the ultimate panacea for the finite-sample problems that exists in autoregressive models.

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Figure 1: Various correction factors in the AR(1) model with possible intercept and trend. The solid line is the Bartlett factor as shown in Theorem 1-3 of the paper. The gray line is the extended approximation discussed in Section 5. The dotted line is based on simulation of $\mathbb{E}[LR]$ (top), $\mathbb{E}[LR^c]$ (middle) or $\mathbb{E}[LR^\tau]$ (bottom) and the dashed line is based on a simulated correction factor that leads to a LR test with 5% size.

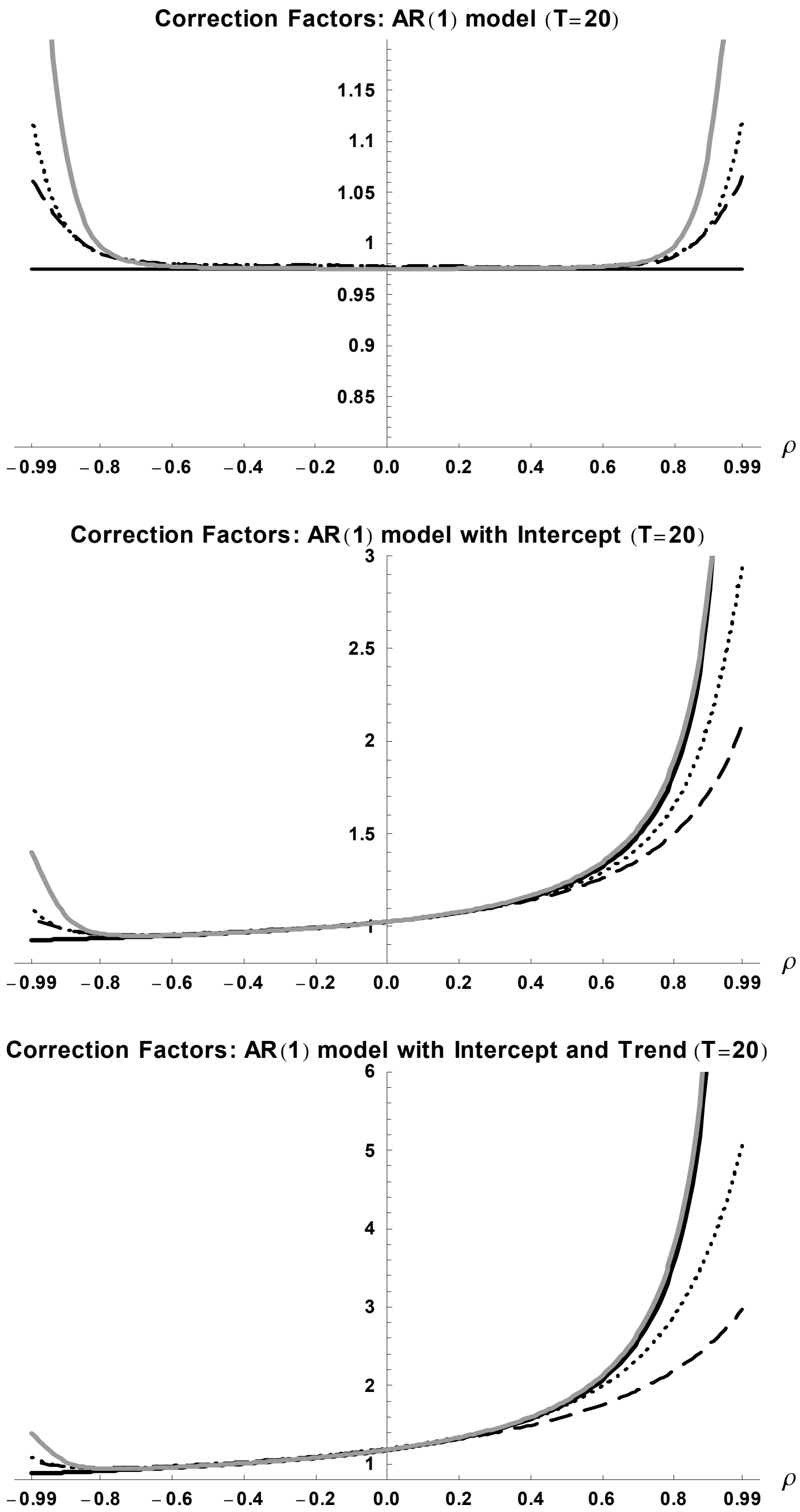

Figure 2: Rejection frequencies in the AR(1) model with possible intercept and trend. The dashed gray line is based on the LR statistic and the asymptotic χ^2-distribution, while the black line is based on the Bartlett-corrected LR statistic using the adjustments shown in Theorem 1-3. The dotted line uses the simulated value of $\mathbb{E}[LR]$ (top), $\mathbb{E}[LR^c]$ (middle) or $\mathbb{E}[LR^\tau]$ (bottom) as correction factor.