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Abstract

The begin and end dates of cartels are often ambiguous, despite competition authorities stating them with precision. The legally established infringement period(s), based on documentary evidence, need not coincide with the period(s) of actual cartel effects. In this paper, we show that misdating cartel effects leads to a (weak) overestimation of but-for prices and an underestimation of overcharges. Total overcharges based on comparing but-for prices to *actual* prices are a (weak) underestimation of the true amount overcharged, irrespective of the type and size of the misdating. The bias in antitrust damage estimation based on *predicted* cartel prices can have either sign. We extend the before-during-and-after method with an empirical cartel dating procedure that uses multiple structural break tests to determine the actual begin and end date(s) of the effects of collusive agreements. Empirical findings in the European *Sodium Chlorate* cartel corroborate our theoretical results.

JEL-codes: C22, C51, L41.

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1 Introduction

Collusive practices such as price-fixing and market sharing allow firms to exert market power they would otherwise not have, artificially restrict output and increase prices for

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longer periods of time. Cartels are per se illegal, yet competition authorities around the world regularly catch and convict some. A company found participating in collusion can expect penalties that are principally based on the value of its sales affected by the illegal agreements. In addition, customers that purchased at overcharged prices as a result of the cartel conspiracy can seek reparation of the antitrust damages they sustained in civil litigation against any of the cartel members.

To determine the quantum of affected commerce and antitrust damages requires knowing with some precision when and how long the cartel was effective. In their prosecutions of cartel offenses, antitrust agencies specify the begin and end dates of cartel violations with seeming precision, often to the day. These formal cartel dates are typically based on documentary evidence of communication, such as minutes of the initiating cartel meeting or e-mails in which members disassociate themselves from the agreements. They need not coincide with the point(s) in time at which the anticompetitive effects produced by the cartel violation began and ended, nor with the actual period(s) of collusion.

Effective and formal cartel dates can differ by the nature of the conspiracy or industry. Collusive agreements can concern future developments, such as planned price rises, the steady dismantling of capacity, or postponement of innovation. Cartels often form and destabilize gradually over time, as members initially join and ultimately defect sequentially. Periods of high prices can sometimes have been temporarily interrupted by price wars, punishments or internal tensions from which the collusion regathered. The cartel may have become ineffective, for example due to the rise of competitive imports, long before its members ultimately disband. Medium-term supply contracts or unsophisticated buyers make a cartel's effects last well after its members parted. In addition, former cartelists may unilaterally keep post-cartel price above competitive levels after the cartel broke down, in order to mask their conspiracy.¹

The formal cartel dates also need not concur with the actual cartel period(s). As collusive restrictions are punishable by object, agencies are under no obligation to determine the effects a cartel may have had empirically. The cartel's formal end is often set at the date of the raid. Cartel periods are obtained in notifications, guilty pleas and consent agreements. While defendants would push for shorter infringement periods, leniency applicants would try to stretch the cover of their amnesty. Agencies may seek to establish the widest possible actual period as a "single and continuous" infringement.² They will pass up, however, on weaker founded suspicion of earlier talks or later meetings that might not

¹See Harrington (2004).

²The legal classification "single and continuous infringement" is often used by the European Commission in its cartel decisions for treating the continuum of collective activity from the meeting of minds in a cartel as one violation. It does not necessarily mean there were no intermittent periods of competition. See Joshua (2009).

withstand challenge on appeal.³ In settlements, the cartel period is subject to negotiation and may therefore be set shorter than the evidence would permit. In initiating damages claims, private plaintiffs may need to choose the class period too early in the litigation to be able to date it sufficiently carefully.

The American Bar Association indeed warns in its manual *Proving Antitrust Damages*:

“[T]he beginning or end of the effects of the alleged unlawful conduct may not coincide with the beginning or end of the conduct itself. The effect might occur later, end earlier, or last longer than the conduct. Experts should rely on the evidence developed in discovery, market facts, and the analysis of liability experts when determining the relevant starting and ending dates for calculating damages.”⁴

Similarly does the European Commission recognize in its *Practical Guide* that:

“Some infringements start, or cease, gradually: and often doubt exists regarding the exact beginning of an infringement and, in particular, the effects it produces. (...) Econometric analysis of observed data can be a way to identify when the infringement’s effects started or ceased.”⁵

In this paper, we study the consequences for antitrust damage estimation of using cartel dates that are different from the actual begin and end dates of the effects of the collusion. Obviously, different dates imply different periods and volumes purchased over which damages can be claimed—possibly more. We show that misdating the period(s) in which a cartel was effective introduces a bias that leads to a (weak) overestimation of but-for prices and an underestimation of overcharges. The effect on total damages depends crucially on whether overcharges are based on predicted or actual cartel prices. When the effective cartel dates are a priori unknown, the common approach of comparing but-for prices to predicted values is shown to be unreliable. The use of observed prices, on the other hand, leads to a (weak) underestimation of cartel damages, irrespective of the type and size of the misdating error.

³Commission of the European Communities (2013), footnote 38: “It is possible that a competition authority limits the finding of an infringement to a certain period, while in fact the infringement may have had a longer duration.” In its infringement decisions, the Commission often determines that the addressees entered into agreements and concerted practice “... from *at least* dd-mm-yyyy and until *at least* dd-mm-yyyy,” thus leaving possibility for a wider cartel period. In some cases, the Commission explicitly states suspicion of a longer duration than it was able to prove in the text.

⁴American Bar Association (2014), page 318.

⁵European Commission (2013), recital 43.

To overcome bias in damage estimation due to incorrect cartel dates, we propose an empirical procedure for determining the effective cartel period(s) in price time-series that is based on econometric tests for multiple structural change as developed by Bai and Perron (1998, 2003). The method is applied to the European *Sodium Chlorate* cartel. While the cartel became effective shortly after its formal begin date, we find that it was effective long after the formal cartel end date. The empirical findings corroborate our theoretical results. The actual total cartel damage is underestimated by more than 25% when using the formal dates.

Most cartel damage studies disclosed rely on formal cartel dates.⁶ In the few cases in which the period of cartel effects were disputed, methods used for arguing the effective cartel dates vary. White (2001) concludes a shorter effective period of the *Lysine Cartel* than alleged by inspection—as well as a higher but-for price. Marshall, Marx and Raiff (2008) observes that the *Vitamins Cartel* potentially affected prices beyond the formal cartel periods and analyzes a number of alternative effective dates. Davis and Garces (2010) suggests to analyse multiple possible starting and finishing dates, but offers no methods for doing this. Hüscherlath *et al.* (2013) models the transition from cartel to non-cartel regime in the German cement cartel with dummy fractions.

While structural break tests are widely used in economics, published applications in determining cartel effects are few. Carlton and Leonard (2004) reports on an application of Bai-Perron tests on a plaintiff’s price-series in an undisclosed case in which the authors acted for the defense. Having found breaks at all in only a small percentage of runs of Monte Carlo simulations, the authors conclude no statistically significant effect from the alleged conspiracy, and assert that the plaintiffs had determined their damages period by “cherry picking” the time-series for price declines. For the purpose of detecting suspicion of collusion, Harrington (2008) proposes the Quandt (1960)-Andrews (1993) test for a single unknown break date. Crede (2015) explores the Bai-Perron tests also as part of an empirical cartel screen.

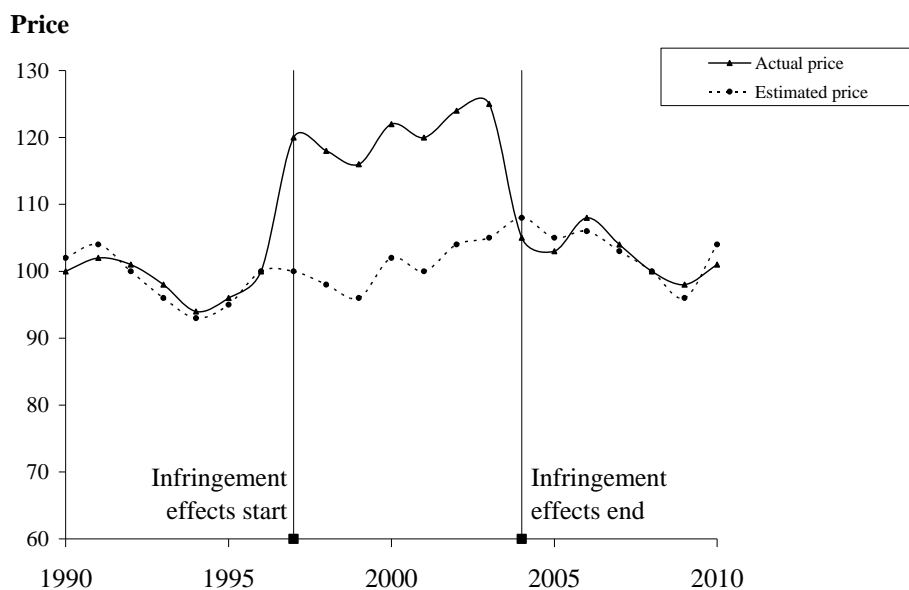
This paper is organized as follows. In Section 2 we review common time-series regression analysis to estimate but-for prices and overcharges. We analyze in detail under which assumptions unbiased cartel damage estimation occurs. Section 3 sets out how using formal cartel dates that are different from the effective cartel dates affects the damage estimates. In Section 4 we describe the empirical cartel dating procedure. Section 5 illustrates the effects of misdating bias by the European *Sodium Chlorate* cartel. Section 6 concludes. The proofs are given in the appendix.

⁶See Bernheim (2008), Nieberding (2006), and Connor and Bolotova (2006).

2 Estimating Cartel Damages using Time Series

Statistical inferences on antitrust damages in empirical cartel studies often rely on time-series analysis, comparing prices over time on the same market, where outside the alleged cartel period the industry is assumed to have been in its normal form of competition. Figure 1 is an illustration of the estimation of cartel effects in a hypothetical case in the European Commission’s 2013 *Practical Guide* on quantifying harm in actions for antitrust damages.⁷

Figure 1: Cartel price effects in the European Commission’s *Practical Guide*.



The solid line (labeled Actual price) is distinctly elevated between the marked start and end of the infringement effect. The dotted line (Estimated price) is what would have been prices under competition. Typically these but-for prices are estimated using multivariate regression models that include the relevant factors affecting product prices. Outside the cartel period, the estimated prices quite closely track actual prices, lending the underlying method of estimation credit as a predictor of what would have been transaction prices during the cartel period, had there not been a cartel. The difference between the cartel price and estimated but-for price is the price overcharge by the cartel. A claimant’s total amount overcharged is the sum over all periods in which the cartel was effective, of each period’s price overcharge multiplied with the total quantity purchased during that period.

For determining but-for prices, the *Practical Guide* presents two regression approaches. In the forecasting approach, competitive prices during the cartel periods are estimated

⁷Commission of the European Communities (2013), page 29.

only from data outside those periods. The alternative before-during-and-after method also involves data from during the cartel periods by including a dummy variable in the regression that quantifies the cartel occurrence as a price shift. The dummy variable approach has the advantage of using (much) more of the available data, potentially making estimates more accurate. However, when the cartel also affected other variables, such as capacity, or when cost pass-through is structurally different under collusion than competition, the price shifts may not reflect this properly.⁸ Tainted explanatory variables can be avoided, however, or corrected to but-for values. Moreover, the dummy variable approach is inherent to cartel dating using structural break tests.

Many recent cartel studies make use of dynamic regression models to quantify cartel overcharges.⁹ It is natural and advantageous to include lagged prices as additional explanatory variables. Parametrically modelling short-run dynamics allows for the typical gradual price adjustments over multiple periods and transition of non-cartel to cartel states. Cartel talks often aim at maintaining or increasing prices relative to recent values leading to serial correlation in prices. Also the lagged price terms contain valuable information on relevant other, but unobserved price determinants. Moreover, whether lagged prices are indeed predictors of cartel prices, regression results will reveal.

In this paper, we take the dummy variable approach to a dynamic regression model. Suppose that price over time develops according to the data generating process (DGP)

$$p_t = \alpha_1 + \alpha_2 D_t + \beta' x_t + \gamma p_{t-1} + \varepsilon_t, \quad t = 1, \dots, T, \quad (1)$$

where p_t is the product unit price in period t , D_t the cartel dummy variable, x_t a set of explanatory variables, p_{t-1} the lagged value of price and ε_t an error term. Collusion is aimed at increasing prices, i.e. $\alpha_2 > 0$. Typical explanatory variables x_t that control for regular changes in prices include cost factors, and demand and supply shifters. The regressor p_{t-1} captures the partial adjustment in prices due to recent changes in the cost structure and market conditions. We assume $0 < \gamma < 1$, i.e. the dynamic relation is stable. Note that DGP (1) is a general reduced form equation of the price that encompasses various specifications with (and without) autoregressive distributed lags.¹⁰

The sample period is $\mathbb{T} = \{1, \dots, T\}$, which can be divided into periods with and without cartel effects, labeled \mathbb{T}_C and \mathbb{T}_N respectively, with $\mathbb{T} = \mathbb{T}_C \cup \mathbb{T}_N$. The presence

⁸See Nieberding (2006), White *et al.* (2006), Marshall and Marx (2012) and McCrary and Rubinfeld (2014).

⁹See White *et al.* (2006), Nieberding (2006), Bernheim (2008) and Davis and Garces (2010).

¹⁰Including the specifications used in the earlier empirical cartel studies referred to in footnotes 8 and 9. Without loss of generality, we limit ourselves in the text to one lagged value of the dependent variable regressor only. The vector of explanatory variables x_t can include both contemporaneous and lagged regressors.

of the cartel effect is quantified by the dummy variable D_t , which is 1 in effective cartel periods ($t \in \mathbb{T}_C$) and 0 otherwise ($t \in \mathbb{T}_N$). Without loss of generality, we consider one single and continuous period of cartel effects with a known begin date T_B and end date T_E , with $1 < T_B < T_E < T$, so that $\mathbb{T}_C = \{T_B+1, \dots, T_E\}$ and $\mathbb{T}_N = \{1, \dots, T_B, T_E+1, \dots, T\}$ and:

$$\left. \begin{aligned} D_t &= 0, & t &\leq T_B \\ D_t &= 1, & T_B < t \leq T_E \\ D_t &= 0, & t &> T_E \end{aligned} \right\}. \quad (2)$$

The coefficient α_2 measures the immediate or short-run response of the price to the collusion as a price level shift.

The but-for price in period t is the expected price level in t when the cartel dummy in (1) is switched off ($D_t = 0$), or

$$bfp_t = \alpha_1 + \beta' x_t + \gamma bfp_{t-1} + \varepsilon_t, \quad t \in \mathbb{T}_C. \quad (3)$$

Outside effective cartel periods, the but-for price and actual price coincide. Within effective cartel periods, the actual price minus the but-for price in each period is that period's overcharge. Noting that $bfp_{T_B} = p_{T_B}$, we can infer from (1) and (3) the per period overcharge

$$O_t = p_t - bfp_t = \frac{1 - \gamma^{t-T_B}}{1 - \gamma} \alpha_2, \quad t \in \mathbb{T}_C. \quad (4)$$

Typically, the basis for a cartel damages claim is the total amount overcharged, defined (before interest) as

$$CD = \sum_{t=T_B+1}^{T_E} O_t Q_t, \quad (5)$$

in which Q_t is the actual quantity purchased in period t .

We analyze whether population magnitudes such as the but-for price (3), overcharge (4) and total cartel damages (5) are accurately estimated based on a sample of time-series observations (p_t, x'_t, D_t) , combined with per period quantities purchased Q_t , $t = 1, \dots, T$. The possible bias in these estimators can be characterized in terms their probability limit, which is the value around which their distribution is increasingly concentrated as the sample size increases. In specific situations, the probability limit will coincide with the mean of the estimators. However, in general this expected value cannot be derived analytically. For its approximation by the probability limit to be well defined, we will consider the estimation of averages per period—rather than the total amount, which does not converge as T increases. In the following, we assume that as the sample size increases, the number of observations in both the cartel and the non-cartel period increases proportionally, which is customary in the econometric literature on structural change.¹¹

¹¹See Perron (1989).

The probability limit of the average overcharge over the cartel period

$$\bar{O} = \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} O_t, \quad (6)$$

can be shown to be equal to

$$\text{plim } \bar{O} = \frac{\alpha_2}{1 - \gamma}. \quad (7)$$

That is, DGP (1) implies an immediate shift of magnitude α_2 in the level of the product price as a result of the collusive agreement, controlling for usual developments in price caused by the explanatory variables in x_t . Due to the autoregressive short-run dynamics, this initial effect accumulates by gradual price adjustment over time to the full cartel effect of size $\alpha_2/(1 - \gamma) > 0$.

The effects of the various explanatory variables on the price in model (1) are estimated by the standard Ordinary Least Squares (OLS) method. Consistent OLS estimation of the regression coefficients is feasible under the following assumption.

Assumption 1: *DGP (1) satisfies:*

$$E[\varepsilon_t | D_{t-j}, x_{t-j}, p_{t-j-1}, j = 0, 1, \dots] = 0, \quad \text{Var}(\varepsilon_t) = \sigma^2,$$

and the random variables p_t and x_t are jointly stationary in deviation from their time-varying means.

For consistency, it is only required that the contemporaneous error term ε_t is uncorrelated with current values of D_t and x_t . Lagged feedback of price to its determinants is allowed for.¹² The begin of the cartel effect, for example, need not be exogenous but may be induced by a recent price fall.¹³ The assumption also implies that $E[\varepsilon_t | \varepsilon_{t-j}] = 0$ and hence $\text{Cov}(\varepsilon_t, \varepsilon_{t-j}) = 0$ for $j > 0$.

Once the coefficients of model (1) have been consistently estimated, but-for prices per period are predicted by their own lagged values, plus current empirical values of the additional covariates

$$\widehat{bfp}_t = \hat{\alpha}_1 + \hat{\beta}' x_t + \hat{\gamma} \widehat{bfp}_{t-1}, \quad t \in \mathbb{T}_C. \quad (8)$$

¹²Distribution theory is markedly different in the case of cointegration between price and its determinants, which we here exclude. Unreported findings, both analytical and by Monte Carlo simulation, suggest that similar conclusions apply also for nonstationary versions of DGP (1).

¹³Note that while it is important to avoid explanatory variables that are affected by the cartel whenever possible, as stressed in White *et al.* (2006) and Marshall and Marx (2012), including them does not invalidate the consistent estimation of cartel effects and dates. The main complication from having collinearity between elements in x_t and D_t is in the construction of the but-for price for p_t , which will then also depend on but-for price paths of those explanatory variables affected by the cartel.

In the recursive dynamic simulation, the natural initial but-for price is the product unit price in the last period of competition before the start of the cartel, i.e. $\widehat{bfp}_{T_B} = p_{T_B}$.

Obviously, the estimated and actual but-for prices generally vary over time. We obtain that the but-for price estimator is consistent, however.

Proposition 1: *Under Assumption 1,*

$$\text{plim} \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} \widehat{bfp}_t = \text{plim} \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} bfp_t = E[bfp_t].$$

Given but-for prices, there are two different practical approaches to the estimation of overcharges in the literature. One is to compare but-for prices as in (8) to *predicted* cartel prices

$$\hat{p}_t = \hat{\alpha}_1 + \hat{\alpha}_2 + \hat{\beta}'x_t + \hat{\gamma}\hat{p}_{t-1}, \quad t \in \mathbb{T}_C, \quad (9)$$

constructed using recursive dynamic simulation with $\hat{p}_{T_B} = p_{T_B}$. They are predictions of the price in case of a cartel ($D_t = 1$) and compared with those of the counterfactual competitive regime ($D_t = 0$).¹⁴ The overcharge in period $s = 1, \dots, T_E - T_B$ during the cartel is then defined as

$$\begin{aligned} \widehat{O}_{1,T_B+s} &= \hat{p}_{T_B+s} - \widehat{bfp}_{T_B+s} \\ &= \hat{\alpha}_2 + \hat{\gamma} \left(\hat{p}_{T_B+s-1} - \widehat{bfp}_{T_B+s-1} \right) \\ &= \frac{1 - \hat{\gamma}^s}{1 - \hat{\gamma}} \hat{\alpha}_2. \end{aligned} \quad (10)$$

The other approach is to compare estimated but-for prices as in (8) to *observed* prices p_t .¹⁵ This defines the overcharge in period s during the cartel as

$$\widehat{O}_{2,T_B+s} = p_{T_B+s} - \widehat{bfp}_{T_B+s}. \quad (11)$$

A crucial difference between the two overcharge types for further damage assessment is that \widehat{O}_1 is based on the constant cartel price effect α_2 —as is the actual overcharge—in DGP (1), whereas \widehat{O}_2 is period-specific.¹⁶

¹⁴This approach to calculating overcharges is taken, for example, in Nieberding (2006). For static models it reduces to the approach of Davis and Garces (2010), McCrary and Rubinfeld (2014) and Laitenberger and Smuda (2015).

¹⁵This approach to calculating overcharges is taken, for example, in Finkelstein and Levenbach (1983) and Harrington (2004). Note that overcharge (11) is also commonly used in the forecasting approach, in which but-for prices are estimated using data from non-cartel periods only.

¹⁶It is not clear which overcharge type is advocated in the Commission's *Practical Guide*. While in various places overcharges are defined, as in \widehat{O}_2 , in comparison to the "price actually paid" (recital 79) and

For correctly specified models, for which Assumption 1 holds and OLS estimators are consistent, there is no difference in expectation for the estimated overcharges. Under consistent estimation of the model parameters we have that

$$\text{plim} \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} \hat{\varepsilon}_t = 0. \quad (12)$$

Hence, average estimated overcharges defined as

$$\bar{O}_j = \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} \hat{O}_{jt}, \quad j = 1, 2, \quad (13)$$

converge to the average actual overcharge as defined in (7), as stated in the following proposition.

Proposition 2: *Under Assumption 1,*

$$\text{plim} \bar{O}_1 = \text{plim} \bar{O}_2 = \text{plim} \bar{O} = \frac{\alpha_2}{1 - \gamma}.$$

For either of the two different approaches to estimating overcharges, the total amount overcharged (5) can be estimated by

$$\widehat{CD}_j = \sum_{t=T_B+1}^{T_E} \hat{O}_{jt} Q_t, \quad j = 1, 2. \quad (14)$$

In order to analyze the accuracy of these estimators, the relation between quantities and prices needs to be made explicit. Let the per period quantity purchased, Q_t , $t = 1, \dots, T$, be price-dependent and stochastic with

$$\left. \begin{aligned} E[Q_t] &= Q_C, & t \in \mathbb{T}_C, \\ E[Q_t] &= Q_N, & t \in \mathbb{T}_N. \end{aligned} \right\} \quad (15)$$

We then make the following assumption.

Assumption 2: *Quantities Q_t satisfy:*

(i) $Q_N \geq Q_C$; and

(ii) $E[Q_t \varepsilon_{t-j}] \leq 0$, $t = 1, \dots, T$, $j \geq 0$.

“the observed prices” (recital 101), the *Practical Guide* appears also to interpret the before-during-and-after method as generating a constant overcharge, which is more in line with overcharge definition \hat{O}_1 —see, for example, footnote 77.

Assumption 2 (i) is the natural condition that in periods in which the cartel raised prices, the average quantity purchased will be less than or equal to the volume purchased under the lower competitive prices, i.e., that demand is (weakly) downward sloping. Assumption 2 (ii) is a high-level assumption. It asks that Q_t is either negatively correlated or uncorrelated with current and past price shocks $\varepsilon_{t-j}, j \geq 0$. This is consistent with a (weak) negative correlation between price and quantity. Assumption 2 (ii) is satisfied under specific (sufficient) conditions that depend on the structure of demand. It holds (with equality), for example, when demand is linear and (weakly) downward sloping demand and x_t is strictly exogenous.¹⁷ For general (weakly) downward sloping demand, Assumption 2 (ii) requires stronger independence conditions.

We are now ready for the first result on the damages estimators.

Theorem 1: *Under Assumptions 1 and 2,*

$$\text{plim} \frac{1}{T} \widehat{CD}_1 = \text{plim} \frac{1}{T} CD,$$

and

$$\text{plim} \frac{1}{T} \widehat{CD}_2 \leq \text{plim} \frac{1}{T} CD,$$

with strict equality if and only if $E [Q_t \varepsilon_{t-j}] = 0$.

The use of \widehat{O}_{1t} as overcharge estimator returns a consistent damage estimator, irrespective of the structure of demand, as overcharge definition (10) is independent of quantities in all its components. Overcharge estimator \widehat{O}_{2t} on the other hand does depend on Q_t , and therefore on the period-to-period interaction between price and quantity, through the OLS estimates of the error term. Only when price and quantity are uncorrelated is \widehat{CD}_2 a consistent estimator of CD . For regular demand, their correlation will be negative, implying that \widehat{CD}_2 underestimates true total overcharges. The overcharge method based on actual cartel prices therefore is conservative.

3 Misdating Bias

In this section we analyze the consequences of using a classification of cartel begin and end dates that is different from the actual dates that define the periods \mathbb{T}_C and \mathbb{T}_N . When effective cartel and non-cartel periods are misclassified, the difference between overcharge calculation based on predicted and actual cartel prices turns out to matter fundamentally for cartel damage estimates as limit result (12) no longer holds.

¹⁷See Appendix A, Assumption 2(ii) under Linear Demand.

In Section 3.1 we first consider the simplest theoretical case of a constant cartel overcharge and no further effects on prices. The bias of the overcharge estimator \widehat{O}_1 is found to can have either sign, so that over- or underestimation of cartel damages can occur. The estimator \widehat{O}_2 , however, robustly remains a conservative basis for calculating antitrust damages when the effective cartel period is misclassified. In Section 3.2 we generalize this result for the full model specification (1), that includes explanatory variables and autoregressive dynamics.

3.1 Basic Cartel Overcharge

Consider the simplified DGP

$$p_t = \alpha_1 + \alpha_2 D_t + \varepsilon_t, \quad t = 1, \dots, T, \quad (16)$$

in which the dummy variable D_t is defined as in (2). Note that this abstraction from DGP (1) results in a simple before-during-after comparison of average prices. The cartel overcharge now is a fixed positive margin α_2 over the competitive price level α_1 . We continue to assume that the regression model (16) is structurally correctly specified, i.e. $E[\varepsilon_t|D_t] = 0$ as in Assumption 1.

Suppose now that the cartel dates used in the estimation are misclassified in the sense that they are different from the effective cartel begin and end dates T_B and T_E —we will refer to these dates as the formal cartel dates. That is, the estimated model for p_t is

$$p_t = \alpha_1 + \alpha_2 d_t + u_t, \quad t = 1, \dots, T, \quad (17)$$

in which the cartel dummy variable d_t takes on the following values:

$$\left. \begin{aligned} d_t &= 0, & t &\leq T_b \\ d_t &= 1, & T_b < t \leq T_e \\ d_t &= 0, & t > T_e \end{aligned} \right\}, \quad (18)$$

where T_b is the formal cartel begin date and T_e the formal cartel end date. The OLS estimators of α_1 and α_2 are:

$$\hat{\alpha}_1 = \bar{p} - \hat{\alpha}_2 \bar{d}, \quad \hat{\alpha}_2 = \frac{s_{dp}}{s_d^2}, \quad (19)$$

where \bar{p} and \bar{d} are sample means, s_{dp} is the sample covariance, and s_d^2 the sample variance.

In this basic setting, there are four possible misdating scenarios for the break dates:

$$\left. \begin{aligned} \text{Case 1: } & T_b < T_B, T_e < T_E \\ \text{Case 2: } & T_b < T_B, T_e > T_E \\ \text{Case 3: } & T_b > T_B, T_e < T_E \\ \text{Case 4: } & T_b > T_B, T_e > T_E \end{aligned} \right\}. \quad (20)$$

In Case 1 the cartel is formally dated to begin and end too early—including a formal cartel period that entirely precedes the cartel’s effects. In Case 2 the formal cartel period encompasses the effective cartel period and includes non-cartel periods too. Case 3 is a legally too narrowly defined period, and Case 4 is the mirror image of Case 1.

Obviously, if $T_b = T_B$ and $T_e = T_E$, the OLS estimators are unbiased and consistent, i.e. $\text{plim } \hat{\alpha}_1 = \alpha_1$ and $\text{plim } \hat{\alpha}_2 = \alpha_2$. To analyze the effects of using formal cartel dates that are different from effective cartel dates, we adapt the approach from Perron (1989) on neglected structural breaks to the more general framework of mismeasured structural change. We obtain the following result on the OLS coefficient estimators of model (17) when the DGP is (16).

Lemma 1: *Let $T_b \neq T_B$ and/or $T_e \neq T_E$. Under Assumption 1:*

$$\begin{aligned} \text{plim } \hat{\alpha}_1 &= \alpha_1, & T_b < T_B, T_e > T_E, \\ \text{plim } \hat{\alpha}_1 &> \alpha_1, & \textit{otherwise}, \\ \text{plim } \hat{\alpha}_2 &< \alpha_2, & \forall T_B, T_E, T_b, T_e. \end{aligned}$$

Misdating the cartel dummy regressor d_t in equation (17) leads to an attenuation bias: the estimator of the dummy coefficient α_2 is biased downward, irrespective of the direction of the misdating of the effective cartel dates. In addition, misdating implies that the estimator of the competitive price level α_1 is biased upwards, except when the formal cartel dates (T_b, T_e) encompass the effective dates (T_B, T_E) , in which (only) case the estimator is consistent.

To see why Lemma 1 holds, let $\bar{p}_C = \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} p_t$ be the mean price over the formal cartel period and $\bar{p}_N = \frac{1}{T - (T_e - T_b)} \left(\sum_{t=1}^{T_b} p_t + \sum_{t=T_e+1}^T p_t \right)$ be the mean price over the formal non-cartel periods. It is easily verified that the OLS estimators (19) can be expressed as

$$\hat{\alpha}_1 = \bar{p}_N, \quad \hat{\alpha}_2 = \bar{p}_C - \bar{p}_N. \quad (21)$$

Misspecifying the effective cartel dates amounts to falsely labelling part of the competitive prices as collusive and/or part of the collusive prices as competitive. So including higher than competitive prices in \bar{p}_N results in an upward bias in estimating α_1 , the competitive price level, because \bar{p}_N overestimates the true mean price in the non-cartel period. Only when $T_b < T_B$ and $T_e > T_B$ (Case 2), is $\hat{\alpha}_1$ based on non-cartel data only, so that it is unbiased.

For the same reason, mistakenly including competitive prices in the cartel sample decreases the average \bar{p}_C . This is the case in all misdating scenarios but Case 3, when the

effective cartel period includes the formal cartel period and \bar{p}_C equals the true mean cartel price. As the estimated cartel effect $\hat{\alpha}_2$ is the difference in average price between cartel and non-cartel periods, it has a downward bias in all misdating cases. The attenuation bias in α_2 is persistent in all cases, as either \bar{p}_C or \bar{p}_N is affected, or both. When \bar{p}_C is correct (Case 3), \bar{p}_N is higher than the true mean non-cartel price, and when \bar{p}_N is correct, \bar{p}_C is lower than the true mean cartel price (Case 2). In the other two scenarios \bar{p}_N is too low and \bar{p}_C too high.

Lemma 1 implies that estimators of but-for prices and overcharges are biased too. For this model, actual and estimated but-for prices are defined as $bf p_t = \alpha_1 + \varepsilon_t$ and $\widehat{bf p}_t = \hat{\alpha}_1 d_t + p_t (1 - d_t)$ respectively. The mean of the but-for price is simply $E[bf p_t] = \alpha_1$. The actual overcharge amounts to $\alpha_2 D_t$, while the estimators are $\widehat{O}_{1t} = \hat{\alpha}_2 d_t$ and $\widehat{O}_{2t} = (p_t - \hat{\alpha}_1) d_t$. The average actual overcharge is therefore $\bar{O} = \alpha_2$, while average estimated overcharges are:

$$\bar{O}_1 = \hat{\alpha}_2, \quad (22)$$

$$\bar{O}_2 = \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} (p_t - \hat{\alpha}_1). \quad (23)$$

We obtain the following result for average estimated but-for prices, and estimated overcharges.

Proposition 3: *Under the conditions of Lemma 1:*

(i)

$$\begin{aligned} \text{plim} \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} \widehat{bf p}_t &= E[bf p_t], \quad T_b < T_B, T_e > T_E, \\ \text{plim} \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} \widehat{bf p}_t &> E[bf p_t], \quad \text{otherwise.} \end{aligned}$$

(ii)

$$\begin{aligned} \text{plim} \bar{O}_1 &< \bar{O}, \\ \text{plim} \bar{O}_2 &< \bar{O}. \end{aligned}$$

Using cartel dates in the before-during-and-after method that are different from the effective cartel dates results in estimated but-for prices that are on average higher than the actual but-for price, except when the formal cartel period encompasses the effective cartel

period, in which case they are correct on average. Note that while on average but-for prices are higher, in individual time periods but-for prices can be lower using formal dates. In addition, Proposition 3 shows that the average overcharge is strictly underestimated for all types of misdating.

The effects that misclassifying the effective cartel period has on total damages is twofold, since the total overcharge is determined by the period overcharge(s), as well as by the number of periods designated as collusive, with a quantity purchased each. The overcharge definition used, \widehat{O}_{1t} or \widehat{O}_{2t} , turns out to matter materially for the consequences of misdating, as shown in the following result.

Theorem 2: *Under the conditions of Lemma 1 and Assumption 2:*

(i)

$$\begin{aligned} \text{plim } \frac{1}{T} \widehat{CD}_1 &\geq \text{plim } \frac{1}{T} CD, & T_b < T_B, T_e > T_E, \\ \text{plim } \frac{1}{T} \widehat{CD}_1 &< \text{plim } \frac{1}{T} CD, & T_b > T_B, T_e < T_E, \\ \text{plim } \frac{1}{T} \widehat{CD}_1 &\begin{matrix} \geq \\ \leq \end{matrix} \text{plim } \frac{1}{T} CD, & \textit{otherwise}. \end{aligned}$$

(ii)

$$\begin{aligned} \text{plim } \frac{1}{T} \widehat{CD}_2 &\leq \text{plim } \frac{1}{T} CD, & T_b < T_B, T_e > T_E, \\ \text{plim } \frac{1}{T} \widehat{CD}_2 &< \text{plim } \frac{1}{T} CD, & \textit{otherwise}. \end{aligned}$$

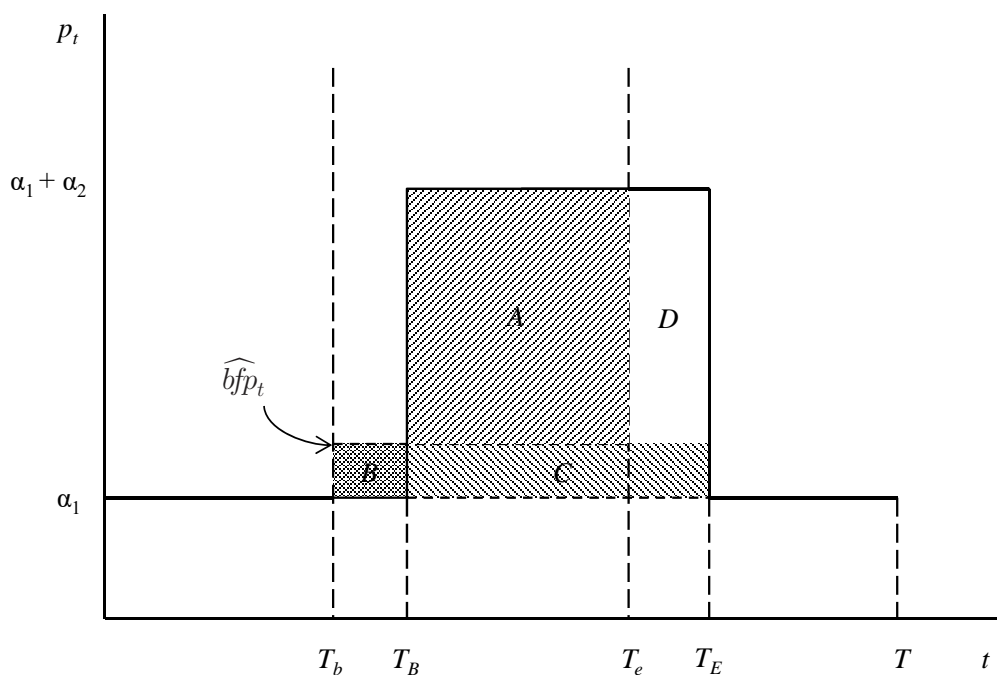
The estimator \widehat{CD}_1 can under- as well as overestimate the actual damage, because the constant overcharge $\widehat{O}_{1t} = \hat{\alpha}_2$, while (weakly) underestimated in all misdating scenarios, is applied to a sum total of quantities labelled as purchased under the cartel regime that may be higher or lower than the actual total quantity.

The estimator \widehat{CD}_2 , on the other hand, for all misdating scenarios provides a conservative estimate, in the sense that the actual total damage during the effective cartel period (T_B, T_E) is at least as large as the damage estimated using misclassified dates in the but-for price estimation. In addition to (weakly) underestimating the actual overcharge, because the overcharge \widehat{O}_{2t} is period-specific, in misdated estimations that lead to higher than actual but-for prices (i.e. in all cases but Case 2) this overcharge becomes negative in competitive periods falsely labelled as collusive (i.e. for $t \in \mathbb{T}_N$, where $\alpha_2 = 0$), thus decreasing the total damage estimate.

When the dates are correct, by Theorem 1, \widehat{CD}_1 is a consistent estimator. Hence all bias in \widehat{CD}_1 established in Theorem 2 is due to misdating. For \widehat{CD}_2 Theorem 1 showed a downward bias already when price and quantity are negatively correlated. The misdating bias in \widehat{CD}_2 is therefore the sum of this bias under correct dating and additional bias due to misdating. Only in Case 2 is the latter effect zero, so that the downward misdating bias in \widehat{CD}_2 is of equal magnitude as the bias established in Theorem 1. In the other three cases, however, the misdating bias is (in absolute value) strictly larger. The relative size of the two sources of bias, however, is case-specific, depending on the own-price elasticity of demand, and the extent of the misdating.

Figure 2 illustrates how the use of estimator \widehat{O}_{2t} as a basis for damage calculation is conservative for Case 1, in which both the begin and end of the cartel are formally dated too early, but the periods still overlap. The actual cartel price increase by α_2 lasts from T_B to T_E . The vertical dashed lines at T_b and T_e are the formal cartel dates used in the estimation. The average actual but-for price is α_1 , yet by using the misclassified formal cartel dates, the estimator $\widehat{\alpha}_1$ uses price data from prior to T_b and past T_e —indicated in Figure 2 with heavier horizontal lines. The estimate therefore includes observations between T_e and T_E that are actually cartel prices, but falsely labelled competitive prices. Inclusion of these higher prices increases the estimated but-for price \widehat{bfp}_t above the level of the actual but-for price.

Figure 2: Theorem 2 illustrated for Case 1.



For three reasons, therefore, the estimated damage total is smaller than the actual damage. First, the overcharges are smaller by area C , because the but-for price is overestimated. Second, the formal cartel end date T_e lies before the effective cartel end date T_E , so that part D is missed. Third, during the period $T_b < T_B$, the overestimation of the but-for price results in negative overcharges the size of area B , which is subtracted from the damage estimate area A .

Note that the longer is the non-cartel period misclassified as cartel period, the higher is the estimated but-for price. In Figure 2, if T_b were further to the left of T_B , \widehat{bfp}_t would be higher—to see this, note that the mean price over the formal cartel period \bar{p}_C decreases, while (for a fixed sample period), the mean price over the formal non-cartel period \bar{p}_N increases. The conservative damage total \widehat{CD}_2 would further decrease, as both the positive damage measured decreases and the area subtracted increases. The effect of this larger misdating error on \widehat{CD}_1 , however, is ambiguous. While \widehat{O}_{1t} is lower, the earlier formal cartel begin date brings more of the larger competitive volume under the damage period. These effects trade off, depending among other things on the own-price elasticity of demand.

Case 4 is the mirror image of the analysis above. In Case 3, in which the formal cartel period falls entirely within the effective period, there are no negative overcharges, yet the but-for price is grossly overestimated and two periods of actual damages left and right from the effective cartel dates are not counted. Only when the formal period includes the effective period, i.e. $T_b < T_B$ and $T_e > T_E$ (Case 2), in which the actual and estimated but-for prices coincide, will overcharges calculated outside the cartel period be zero on average, thereby not contributing to the damage estimate, so that \widehat{CD}_2 is an unbiased estimator of actual damages. \widehat{CD}_1 , however, may still overestimate the actual damage in Case 2, as \bar{p}_C (and so \widehat{O}_{1t}) decreases in the length of the formal cartel period, while at the same time more purchases are alleged to be affected.

3.2 Full Price Dynamics

Next we consider the general model specification (1). First note that there is no reason to think that estimator \widehat{CD}_1 will improve in the extended model, and so we need not examine it further. To see if \widehat{CD}_2 remains a robust conservative estimator, consider the estimated model

$$p_t = \alpha_1 + \alpha_2 d_t + \beta x_t + \gamma p_{t-1} + u_t, \quad t = 1, \dots, T, \quad (24)$$

to which the four misdating scenarios apply through the dummy d_t . To simplify the analysis, we consider the case of a single explanatory variable x_t in this section.

We obtain the following results on the OLS coefficient estimators of model (24) when the DGP is (1).

Lemma 2: *Let $T_b \neq T_B$ and/or $T_e \neq T_E$. Under Assumption 1:*

$$\begin{aligned} \text{plim } \hat{\alpha}_1 &< \alpha_1, \\ \text{plim } \hat{\alpha}_2 &< \alpha_2, \\ \text{plim } \hat{\beta} &\leq \beta, \quad \text{if } m_{xp-1} \geq 0, \\ \text{plim } \hat{\gamma} &> \gamma, \end{aligned}$$

where $m_{xp-1} = \text{plim } \frac{1}{T} \sum_{t=1}^T x_t p_{t-1}$.

The misdating bias now prevails in all cases throughout. The cartel coefficient α_2 continues to be underestimated in all misdating scenarios. The intercept α_1 now is always underestimated as well. The autoregressive coefficient γ is overestimated always, whereas the direction of the bias in the coefficient β of the explanatory variable depends on the sign of the particular data moment m_{xp-1} .

With bias in the OLS estimators of all regression coefficients, it is not obvious what the impact is of misspecified cartel begin and end dates on the estimated but-for prices, cartel overcharges and total damage. We obtain the following generalization of Proposition 3 for the average but-for price.

Proposition 4: *Under the conditions of Lemma 2:*

(i)

$$\begin{aligned} \text{plim } \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} \widehat{bfp}_t &= \text{plim } \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} bfp_t, \quad T_b < T_B, T_e > T_E, \\ \text{plim } \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} \widehat{bfp}_t &> \text{plim } \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} bfp_t, \quad \text{otherwise.} \end{aligned}$$

(ii)

$$\text{plim } \bar{O}_2 < \text{plim } \bar{O}.$$

Hence, but-for prices remain to be overestimated in all misdating scenarios but with inclusive formal dates, and the average overcharge is underestimated in all. Moreover, Theorem 2 generalizes for the preferred overcharge estimator \widehat{O}_{2t} as well.

Theorem 3: *Under the conditions of Lemma 2 and Assumption 2:*

$$\begin{aligned} \text{plim} \frac{1}{T} \widehat{CD}_2 &\leq \text{plim} \frac{1}{T} CD, & T_b < T_B, T_e > T_E, \\ \text{plim} \frac{1}{T} \widehat{CD}_2 &< \text{plim} \frac{1}{T} CD, & \textit{otherwise}. \end{aligned}$$

We conclude that when the effective cartel dates are unknown, the preferred approach to cartel damages estimation is to compare estimated but-for prices to *observed* prices, not *predicted* cartel prices. The estimator \widehat{CD}_2 applied to the misclassified cartel dates provides a lower bound for the actual cartel damages, irrespective of the type and size of the misdating. \widehat{CD}_2 furthermore is consistent as long as the formal cartel dates used in the estimation encompass the effective cartel dates. In conclusion, when the effective cartel dates are unknown, the preferred approach to cartel damages estimation is to use conservative overcharge estimator \widehat{O}_{2t} .

4 Cartel Dating using Multiple Structural Break Tests

The bias from misspecifying effective cartel begin and end dates can be constrained by supplementing the before-during-and-after method with tests for multiple structural change. Structural break tests seek to establish the break date(s) that best fit the regression to the data, by trying many different divisions of the time-series, each with different candidate dates to switch the dummy variables. We use Bai-Perron test procedures for detecting *a priori* unknown multiple structural breaks in time series. Inclusion of the relevant determinants of price avoids spurious breaks, as apparent structural breaks in the price time series can be confounded with omitted variables. Bai and Perron (1998, 2003) suggest to model any serial correlation parametrically by including lagged regressors.¹⁸ While Bai-Perron tests can also be used to determine possible changes in (some or all) coefficients of the explanatory variables, in line with the cartel literature we treat the difference between cartel and non-cartel regimes as price level shifts.

Consider a general version of DGP (1) with up to m different intercepts

$$p_t = \delta_j + \beta' x_t + \gamma p_{t-1} + \varepsilon_t, \quad t = T_{j-1} + 1, \dots, T_j, \quad (25)$$

for $j = 1, \dots, m + 1$ with $T_0 = 0$ and $T_{m+1} = T$. As the intercepts equal $\delta_1, \dots, \delta_{m+1}$, the model allows for $m + 1$ regimes with unknown break dates T_1, \dots, T_m .¹⁹

¹⁸As an alternative approach to treating potential serial correlation in the time series, Bai and Perron (1998, 2003) suggest to estimate a static model and treat serial correlation non-parametrically by exploiting HAC covariance matrix estimation.

¹⁹Note that in model (1) $m = 2$, implying $\delta_1 = \alpha_1$, $\delta_2 = \alpha_1 + \alpha_2$ and $\delta_3 = \alpha_1$.

Given a set of candidate break dates T_1, \dots, T_m , the least-squares estimates minimize the residual sum of squares from (25)

$$S_T(T_1, \dots, T_m) = \sum_{j=1}^{m+1} \sum_{t=T_{j-1}+1}^{T_j} (p_t - \delta_j - \beta'x_t - \gamma p_{t-1})^2. \quad (26)$$

The effective cartel dates $(\hat{T}_1, \dots, \hat{T}_m)$ are found as

$$(\hat{T}_1, \dots, \hat{T}_m) = \operatorname{argmin}_{T_1, \dots, T_m} S_T(T_1, \dots, T_m), \quad (27)$$

where the minimization is over all partitions for which $T_j - T_{j-1} \geq h$, the trimming parameter h being the minimal number of observations between two breaks.

The null hypothesis of no break is tested against the alternative hypothesis of m breaks, using the standard F -test statistic for goodness of fit of the model with and without the estimated break dates $(\hat{T}_1, \dots, \hat{T}_m)$, developed in Chow (1960) for known break dates. This is referred to as the sup F -test: the maximum over all possible Chow F -tests. It has a non-standard asymptotic distribution, for which asymptotic critical values are provided in Bai and Perron (1998). Perron (2006) conjectures that most tests for structural change will have non-monotonic power when the true number of breaks is larger than the number of breaks imposed in the construction of the test statistic. It is advisable, therefore, to report outcomes of the sup F -test for a number of different choices of m . As the number of breaks in the data is not *a priori* known, use is made of the unweighted and weighted double maximum tests, labeled UD max and WD max, which are the maxima of a series of sup F -statistics, to test the null hypothesis of no break versus an unknown number of breaks—up to some prespecified maximum.

The break dates so found indicate structural changes in the specification, such as would be caused by a regime switch from competition to collusion, and *vice versa*. The procedure thus allows for establishing empirically whether there were effects of the cartel violation and whether they were continuous, or rather consisted of several collusive episodes with competitive intermittants. It is important to note that in the dynamic specification, the breaks in the time series should be interpreted as the dates where the cartel began to become effective, or started to lose its effectiveness. The transitions from competitive to collusive prices and back typically are gradual. For various reasons discussed, price levels may continue to be affected by a cartel long collapsed. The end date of such a “lingering effect” of a cartel would be when the but-for prices and the actual prices have fully converged.²⁰

²⁰In a landmark decision in a paper case in 2010, the German Federal Court of Justice allowed to take “lingering effects” of the cartel into account as a factor augmenting the damage—see German Federal Court of Justice, 28 June 2011, ORWI, recital 84. See also Hüscherlath *et al.* (2013).

A concern about the test procedures is that they rely on asymptotic distribution theory. Monte Carlo simulation studies on univariate time series models have shown that finite sample distributions of the various test statistics may deviate substantially from their asymptotic approximations.²¹ In particular, size distortions are found to grow (and power to decrease) when time series become more persistent.²² In principle, the bootstrap can be used to get a better approximation of the finite sample distribution of the structural break tests. Diebold and Chen (1996) show that for the univariate first-order autoregressive model, using a standard nonparametric bootstrap leads to tests with the correct size.

To the best of our knowledge, no such simulation results exist for asymptotic and bootstrap Bai-Perron tests for multiple regression models with autoregressive short-run dynamics and additional explanatory variables. We have therefore analyzed in a small scale Monte Carlo study for our DGP (1) whether the implementation of structural break tests for detecting and dating cartel effects has enough size control. Results are reported in Appendix B. The actual size of the statistical tests for structural change is found to be close to nominal significance levels when the data are moderately persistent. Size distortions occur, however, when there is high persistence, which could lead to spurious breaks. Yet the bootstrap can effectively deal with this problem and leads to test procedures with the correct size. Therefore, in practice the bootstrap version of the break tests should be favored.

5 Dating the *Sodium Chlorate* Cartel

In June 2008, the European Commission adopted a cartel infringement decision in *Sodium Chlorate* against four groups of chemical producers, imposing fines of 79 million Euro in total.²³ Sodium chlorate is a chemical compound synthesized from the electrolysis of salt and water. It is applied in the bleaching of pulp to make it suitable for the manufacture of printing and tissue grade papers that meet the Elemental Chlorine Free (ECF) paper standard. The cost of sodium chlorate as part of the final chemical pulp prices is about 1%. Supply agreements typically took the form of a medium-term framework contract (usually 3-5 years), including estimated purchase volumes and a clause for price adjustments based on specified cost indicators. Price revisions were made throughout the year and common for the new calendar year.

²¹See Diebold and Chen (1996), O'Reilly and Whelan (2005), and Bai and Perron (2006).

²²Additionally, Bai and Perron (2006) reports Monte Carlo simulation results showing that the use of nonparametric covariance matrix estimators leads to substantially larger size distortions than modeling serial correlation parametrically by using dynamic regression models.

²³European Commission decision of 11/06/2008 in Case COMP/38.695 – *Sodium Chlorate*. The decision was upheld by the European General Court.

When demand began to stagnate in anticipation of the stricter Totally Chlorine Free (TCF) paper standard, and (expected) overcapacity put downward pressure on prices in the first half of the 1990's, the main sodium chlorate producers in Europe formed a cartel to implement a strategy of stabilizing the market. The first cartel meeting of which the Commission obtained evidence was on 21–22 September 1994, in Helsinki, in which Akzo/EKA and Kemira agreed on upward price adjustments for 1995. The cartel expanded in 1995 and 1996, when smaller suppliers joined.

In 1998 some internal tensions are reported to have risen between the cartel members, after one sodium chlorate producer was suspected to have secretly supplied another's customer. Repeated attempts were made to regain cartel stability, leading to coordinated price increases in 1997, 1998 and 1999. In March 2003, Akzo/EKA brought the existence of the cartel to the attention of the European Commission with a successful leniency application. The Commission determined in its decision that the infringement had formally ended February 9th, 2000, when Akzo/EKA had denounced the cartel at a trade association meeting.

The monthly price per ton of sodium chlorate is analyzed in the sample period January 1993 to December 2005 on the basis of $T = 156$ volume-weighted average delivered price observations (p_t) from several large customers of the cartel that together exercised almost half of total European demand for sodium chlorate. Monthly variation in the price series reflects (contractual) price revisions made throughout the year. The price data do not exhibit a clear seasonal pattern.

We estimate the dynamic regression model

$$p_t = \alpha_1 + \alpha_2 d_t + \beta_1 x_{1t} + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 x_{4t} + \gamma_1 p_{t-1} + \gamma_2 p_{t-2} + u_t, \quad (28)$$

in which d is the cartel dummy, x_1 is the electricity price, x_2 labour costs, x_3 Western European chemical pulp production, and x_4 European production capacity for sodium chlorate.²⁴ Electricity and labour are the major input costs in the production of sodium chlorate and the main common contract indexation factors, while pulp production and sodium chlorate production capacity serve as demand and supply shifters respectively. The short-run dynamics of product prices are modelled by two lagged price regressors p_{t-1} and p_{t-2} . Except for the cartel dummy, all variables are measured in logarithms, so that the regression coefficients are (semi-)elasticities.

Table 1 reports the results of various structural break tests. To avoid possible low power due to an underspecified number of breaks, we report sup F -test outcomes for

²⁴Sources are respectively: Eurostat, "Electricity - industrial consumers - half-yearly prices"; OECD System of Unit Labour Cost Indicators; CEPI, European Chemical Pulp Production, 1995Q1 - 2008Q3; Harriman Chemsult, Chemicals Economics Handbook-SRI Consulting and CMAI.

$m = \{1, 2, 3, 4, 5\}$. We use two different trimming parameters $\mu = h/T = \{0.10, 0.15\}$. Furthermore, the UD max statistic is used to determine the number of breaks. For both asymptotic and bootstrap critical values, all sup F -tests as well as the UD max test indicate two, and only two, break points, estimated at January 1995 and February 2002, marking structural changes in the effectiveness of the cartel.

Table 1: Empirical outcomes from structural break tests in Sodium Chlorate.

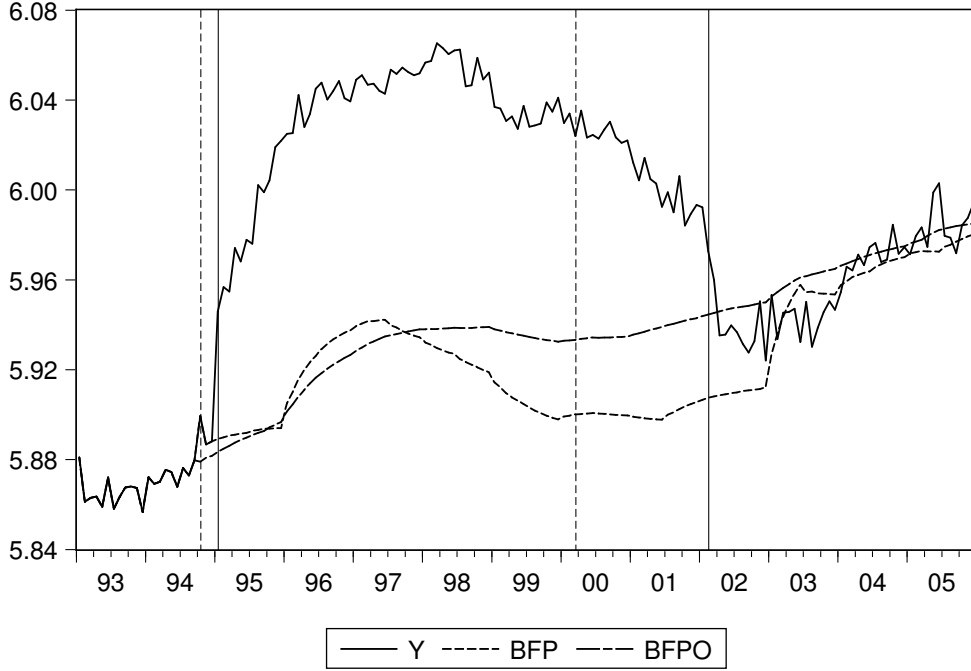
μ	homogeneity		heterogeneity		asymptotic cv		bootstrap cv	
	0.10	0.15	0.10	0.15	0.10	0.15	0.10	0.15
supF(1)	1.36	1.36	1.29	1.29	9.10	8.58	15.27	14.75
supF(2)	27.34	8.44	29.71	10.07	7.92	7.22	12.67	11.81
supF(3)	22.81	3.54	24.24	3.87	6.84	5.96	11.96	10.15
supF(4)	21.01	4.90	21.82	5.49	6.03	4.99	11.45	8.97
supF(5)	20.16	4.04	19.93	4.10	5.37	3.91	10.94	6.81
UD max	27.34	8.44	29.71	10.07	9.52	8.88	16.01	15.35
#breaks	2	0	2	2				
break dates	1995:1		1995:1	1995:1				
	2002:2		2002:2	2002:2				

Note: nominal size is 5%. Bootstrap critical values (cv) are calculated using 1000 replications.²⁵

$\mu = h/T$ is the trimming parameter.

Figure 3 displays the actual sodium chlorate prices as the solid line (labelled Y). The effective cartel dates estimated are indicated with solid vertical lines, while the dashed vertical lines are the formal cartel begin and end dates—September 1994 and February 2000. *Sodium Chlorate* is an example of Case 1 misdating. The effective cartel begin date matches the Commission’s evidence that the initial cartel meetings in September 1994 aimed to raise prices as of the following year: prices increase steeply from the beginning of the year 1995, to peak in 1998. Even though prices subsequently declined in the second half of 1998—in fact with electricity prices, the main cost component in the production of sodium chlorate—they stabilized with the reported resolve of internal stability issues. The effects of the cartel on prices subsequently persisted until the beginning of 2002, when sodium chlorate price fell rapidly, almost two years after the formal end date February 9th, 2000. By the time the existence of the cartel was notified to the European Commission, in March 2003, prices had returned to a more stable competitive trend.

Figure 3: Cartel price effects in *Sodium Chlorate*.



Note: Y is actual price; BFP (BFPO) is but-for price using effective (formal) cartel dates.

We estimate the dynamic model (28) with OLS for two date specifications of the cartel dummy. Using the formal dates, i.e. September 1994 – February 2000, we obtain:

$$\hat{p}_t = 0.783 + 0.013d_t + 0.007x_{1t} + 0.023x_{2t} - 0.005x_{3t} - 0.054x_{4t} + 0.596p_{t-1} + 0.299p_{t-2}. \quad (29)$$

(0.358) (0.003) (0.011) (0.016) (0.019) (0.059) (0.077) (0.074)

While using the effective cartel dates, i.e. January 1995 – February 2002, we find:

$$\hat{p}_t = 2.033 + 0.029D_t + 0.056x_{1t} + 0.031x_{2t} + 0.002x_{3t} - 0.145x_{4t} + 0.455p_{t-1} + 0.300p_{t-2}. \quad (30)$$

(0.369) (0.004) (0.011) (0.013) (0.017) (0.053) (0.074) (0.067)

Using the formal cartel dates returns coefficients of the explanatory variables that are all insignificant, whereas the specification does contain the primary cost factors known to determine price in the sodium chlorate industry. Applying the effective cartel dates instead, coefficients all have the expected sign. Capacity is the most important determinant of price, followed by electricity costs.

The empirical model has strong combined explanatory power, with a large F statistic (827.70) on the overall significance of the regression. For both specifications, the estimated short-run dynamics are adequate, i.e. additional lags of the regressors are not significant. Furthermore, Breusch-Godfrey Lagrange Multiplier misspecification tests do not indicate significant residual autocorrelation, which indicates that all observed autocorrelation in the price has been modelled parametrically by the estimated dynamic specification. The

predictive power of the estimated dynamic model is superior to the static specification. Various sensitivity checks indicate that the estimation results are robust to changes in functional form and sample period.

Comparing the estimates in (29) and (30), their relative magnitude corroborates the results of Lemma 2. The short-run cartel effect $\hat{\alpha}_2$ is more than twice as large using the effective cartel dates rather than the formal cartel dates (2.9% versus 1.3%). The sum of the autoregressive coefficients is also higher under the formal cartel date specification, while the estimated intercept $\hat{\alpha}_1$ is lower.

Using the effective cartel dates specification, but-for prices are constructed by recursive dynamic simulation as

$$\widehat{bfp}_t = 2.033 + 0.455 \widehat{bfp}_{t-1} + 0.300 \widehat{bfp}_{t-2} + 0.056 x_{1t} + 0.031 x_{2t} + 0.002 x_{3t} - 0.145 x_{4t},$$

from January 1995 onwards. The observed prices in November and December 1994 are used as initial values of the period-by-period but-for simulation, i.e. $\widehat{bfp}_{1994:11} = p_{1994:11}$ and $\widehat{bfp}_{1994:12} = p_{1994:12}$. In Figure 3, these are plotted as the dotted line (labelled BFP). While indeed the cartel started to lose its effectiveness in raising price in February 2002, the but-for and the actual price series are fully converged only from February 2003, which identifies the end date of the cartel effect. This lingering effect is consistent with contracts and negotiations common in the industry, as well as the late leniency application.

Similarly, but-for prices using the formal cartel dates are simulated as the dash-dotted line (labelled BFPO). On average, the but-for prices estimated on the basis of the formal dates are 0.95% higher than the but-for prices corresponding to the effective dates, which corroborates Proposition 3(i)—notwithstanding the fact that the formal dates but-for prices are lower than the effective dates but-for prices during the period 1996–1997.

The overcharge estimates are also in conformity with the theoretical predictions. Using the effective dates, we find that the average \widehat{O}_1 and \widehat{O}_2 overcharges are almost equal, corroborating Proposition 2. Moreover, using the formal cartel dates, the average \widehat{O}_1 and \widehat{O}_2 overcharges are respectively 9.33% and 8.02% lower than their counterparts using the effective dates, which is in line with Proposition 3(ii). The downward bias in overcharge estimation is more or less equal for \widehat{O}_1 and \widehat{O}_2 , which is consistent with the finding that under the conditions of Proposition 3 in Case 1, $\text{plim } \overline{O}_1 = \text{plim } \overline{O}_2$.

Total damage estimates \widehat{CD}_1 and \widehat{CD}_2 are almost identical under the effective dates. From Theorem 1, this indicates that sodium chlorate demand has a very low own-price elasticity. Indeed does estimation of a dynamic regression of sodium chlorate quantities on prices show an insignificant price effect. This is consistent with demand for sodium chlorate use in bleaching being driven by mainly by pulp and paper production and prices, the availability of only imperfect and more expensive substitutes, and the cost for sodium

chlorate being a negligible cost factor in chemical pulp production.

The damage estimates \widehat{CD}_1 and \widehat{CD}_2 are respectively 28.32% and 28.57% lower using the formal rather than the effective cartel dates, which is consistent with Theorem 3. These significant differences clearly illustrate the anomalous effects of misspecifying cartel begin and end dates: it results in downward bias that can be large in comparison to proper dating. The bias is due to both lower overcharges and a shorter damage period. The downward bias in damage estimation is equal for \widehat{CD}_1 and \widehat{CD}_2 , which again is consistent with price-insensitive demand in Case 1, for which $\text{plim } \frac{1}{T}\widehat{CD}_1 = \text{plim } \frac{1}{T}\widehat{CD}_2$.

6 Concluding Remarks

We have shown that proper cartel dating is crucial for obtaining accurate estimates of cartel damages. Using misclassified cartel begin and end dates leads to a (weak) overestimation of but-for prices and an underestimation of overcharges. When overcharges are defined as the difference between *actual* prices and but-for prices, the resulting damage estimator is conservative in the sense that it always leads to a (weak) underestimation of the true damage. While a longer formal cartel period subsumes more volume under the damage claim, but-for prices are (weakly) overestimated, so that volumes purchased in falsely alleged cartel period parts are premultiplied by non-positive overcharges. Instead, using the difference between *predicted* cartel prices and but-for prices, both over- and underestimation of cartel damages can occur.

The misdating bias can be avoided with econometric tests for structural change. Even with proper empirical cartel dating though, when the effective cartel dates are unknown the preferred approach to cartel damages estimation is to compare estimated but-for prices to observed prices. The approach is conservative when the effective cartel dates are exactly estimated, and remains so for an error margin. It is more robust against misdating than comparisons with predicted cartel prices, which recent reports appear to favor. The empirical findings in *Sodium Chlorate* corroborate our theoretical results. Using the formal cartel dates estimates damages more than 25% lower.

Under tort law, the injured party is entitled to compensation from the wrongdoer for harm that is caused by—and typically follows—the tortious act. Damages resulting from collusion outside the formal cartel period should therefore in principle be permitted as part of an antitrust claim. Together with qualitative evidence of the cartel’s *modus operandi*, cartel dating can help corroborate or falsify the collusion as the legally recognizable cause of the harm. Plaintiffs in an antitrust damages action should apply dating also to avoid the risk of leaving part of their actual cartel damages unclaimed. Defendants can use the techniques to show an alleged cartel was ineffective, when it was. Agencies may want to

consider estimating effective cartel dates for the purpose of setting deterring fines. They otherwise do good to specify the widest possible formal cartel period the evidence permits, so as to reduce the risk of misdating bias for litigants when cartel dating is not an option.

In line with the cartel literature, and suiting *Sodium Chlorate*, we have developed theory and tests for cartel price level shifts. The empirical cartel dating procedure can be extended to determine breaks in explanatory variables between collusive and competitive regimes too, and thus answer to the criticism that cost-elasticities may be structurally higher in competition than collusion. Ignoring such parameter breaks where they are present could lead to underestimation of the cost coefficients. However, the considered break test procedure does not offer a strategy for discriminating between different parameter breaks, which may occur also at different dates. We leave this for future research.

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Appendix A: Proofs

Proof of Proposition 1. Following the standard asymptotic analysis of structural breaks (Perron, 1989), we assume that $T_B = \lambda_B T$ and $T_E = \lambda_E T$ with the break fractions λ_B and λ_E fixed numbers for all values of T .

We have for the average estimated but-for price

$$\text{plim} \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} \widehat{bfp}_t = \text{plim} \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} bfp_t + \text{plim} \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} (\widehat{bfp}_t - bfp_t).$$

We will analyze the limiting behavior of the second right-hand side term in more detail. Noting that

$$\widehat{bfp}_{T_B} = bfp_{T_B} = p_{T_B},$$

in period $T_B + 1$ we have for the estimated and effective but-for prices:

$$\begin{aligned} \widehat{bfp}_{T_B+1} &= \hat{\gamma} p_{T_B} + \hat{\beta}' x_{T_B+1} + \hat{\alpha}_1, \\ bfp_{T_B+1} &= \gamma p_{T_B} + \beta' x_{T_B+1} + \alpha_1 + \varepsilon_{T_B+1}. \end{aligned}$$

Therefore, we can write for the prediction error

$$\begin{aligned} v_{T_B+1} &= \widehat{bfp}_{T_B+1} - bfp_{T_B+1} \\ &= (\hat{\gamma} - \gamma) p_{T_B} + (\hat{\beta} - \beta)' x_{T_B+1} + \hat{\alpha}_1 - \alpha_1 - \varepsilon_{T_B+1} \\ &= -\varepsilon_{T_B+1} + O_P(T^{-1/2}), \end{aligned}$$

because estimation errors are $O_P(T^{-1/2})$ under Assumption 1 due to standard asymptotic theory. In period $T_B + 2$ we have for the difference in estimated and effective but-for prices

$$\begin{aligned} \widehat{bfp}_{T_B+2} - bfp_{T_B+2} &= \hat{\gamma} \widehat{bfp}_{T_B+1} - \gamma bfp_{T_B+1} + (\hat{\beta} - \beta)' x_{T_B+2} + \hat{\alpha}_1 - \alpha_1 - \varepsilon_{T_B+2} \\ &= \hat{\gamma} v_{T_B+1} + (\hat{\gamma} - \gamma) bfp_{T_B+1} + (\hat{\beta} - \beta)' x_{T_B+2} + \hat{\alpha}_1 - \alpha_1 - \varepsilon_{T_B+2} \\ &= \hat{\gamma} v_{T_B+1} + v_{T_B+2}, \end{aligned}$$

where for the prediction error v_{T_B+2} we have

$$v_{T_B+2} = -\varepsilon_{T_B+2} + O_P(T^{-1/2}).$$

In general, we have for $s = 1, 2, \dots, T_E - T_B$

$$\begin{aligned} \widehat{bfp}_{T_B+s} - bfp_{T_B+s} &= \hat{\gamma}^{s-1} v_{T_B+1} + \hat{\gamma}^{s-2} v_{T_B+2} + \dots + \hat{\gamma} v_{T_B+s-1} + v_{T_B+s} \\ &= \sum_{j=0}^{s-1} \hat{\gamma}^j v_{T_B+s-j} + O_P(T^{-1/2}) \\ &= -\sum_{j=0}^{s-1} \hat{\gamma}^j \varepsilon_{T_B+s-j} + O_P(T^{-1/2}), \end{aligned}$$

where the second line follows from $\hat{\gamma} = \gamma + O_P(T^{-1/2})$. Therefore,

$$\text{plim} \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} \left(\widehat{bfp}_t - bfp_t \right) = - \text{plim} \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} \sum_{j=0}^{s-1} \gamma^j \varepsilon_{T_B+s-j} = 0,$$

by the Law of Large Numbers (LLN) for asymptotically stationary linear processes, using $E[\varepsilon_t] = 0$ and $E[\varepsilon_t \varepsilon_s] = \sigma^2$ for $t = s$, and 0 otherwise. This completes the proof because, due to stationarity (Assumption 1) we have

$$\text{plim} \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} bfp_t = E[bfp_t].$$

□

Proof of Proposition 2. Noting that $bfp_{T_B} = p_{T_B}$ we have for the overcharge

$$O_{T_B+s} = (p_{T_B+s} - bfp_{T_B+s}) = \frac{1 - \gamma^s}{1 - \gamma} \alpha_2,$$

for $s = 1, 2, \dots, T_E - T_B$. Therefore, we have for the average effective overcharge

$$\begin{aligned} \text{plim} \bar{O} &= \text{plim} \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} (p_t - bfp_t) \\ &= \lim \frac{1}{T_E - T_B} \sum_{s=1}^{T_E - T_B} \frac{1 - \gamma^s}{1 - \gamma} \alpha_2 \\ &= \frac{\alpha_2}{1 - \gamma}. \end{aligned}$$

Using \widehat{O}_{1t} we have for the average estimated overcharge

$$\begin{aligned} \text{plim} \bar{O}_1 &= \text{plim} \frac{1}{T_E - T_B} \sum_{s=1}^{T_E - T_B} \frac{1 - \hat{\gamma}^s}{1 - \hat{\gamma}} \hat{\alpha}_2 \\ &= \text{plim} \frac{\hat{\alpha}_2}{1 - \hat{\gamma}} \left(1 - \text{plim} \frac{1}{T_E - T_B} \sum_{s=1}^{T_E - T_B} \hat{\gamma}^s \right) \\ &= \frac{\alpha_2}{1 - \gamma}. \end{aligned}$$

Using \widehat{O}_{2t} we have for the average estimated overcharge

$$\begin{aligned} \text{plim} \bar{O}_2 &= \text{plim} \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} \left(p_t - \widehat{bfp}_t \right) \\ &= \text{plim} \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} (p_t - bfp_t) - \text{plim} \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} \left(\widehat{bfp}_t - bfp_t \right) \\ &= \text{plim} \bar{O}, \end{aligned}$$

which follows directly from Proposition 1 and completes the proof. □

Assumption 2(ii) under Linear Demand. Suppose model (1) and Assumption 1 hold. Consider the stylized linear demand function

$$Q_t = a + bp_t + \epsilon_t,$$

where $a > 0$ and ϵ_t is an error term with $E[\epsilon_t|p_t] = 0$. Then Assumption 2 (ii) holds when demand is downward sloping ($b \leq 0$) and x_t is strictly exogenous, i.e. $E[x_t \epsilon_s] = 0$ for all $s, t = 1, \dots, T$.

From model (1),

$$p_t = \frac{\alpha_1}{1 - \gamma} + \alpha_2 \sum_{i=0}^{\infty} \gamma^i D_{t-i} + \beta \sum_{i=0}^{\infty} \gamma^i x_{t-i} + \sum_{i=0}^{\infty} \gamma^i \epsilon_{t-i},$$

so that

$$Q_t = a + \frac{b\alpha_1}{1 - \gamma} + b\alpha_2 \sum_{i=0}^{\infty} \gamma^i D_{t-i} + b\beta \sum_{i=0}^{\infty} \gamma^i x_{t-i} + b \sum_{i=0}^{\infty} \gamma^i \epsilon_{t-i} + \epsilon_t.$$

Under Assumption 1 and strict exogeneity of x_t ,

$$\begin{aligned} E[Q_t \epsilon_{t-j}] &= E \left[\left(b\beta \sum_{i=0}^{\infty} \gamma^i x_{t-i} + b \sum_{i=0}^{\infty} \gamma^i \epsilon_{t-i} + \epsilon_t \right) \epsilon_{t-j} \right] \\ &= b\gamma^j E[\epsilon_{t-j}^2] \\ &\leq 0, \end{aligned}$$

for $b \leq 0$, with equality if and only if $b = 0$, as $0 < \gamma < 1$. □

Proof of Theorem 1. For the effective damage, we have

$$\begin{aligned} \text{plim} \frac{1}{T} CD &= \text{plim} \frac{1}{T} \sum_{t=T_B+1}^{T_E} (p_t - bp_t) Q_t \\ &= \text{plim} \frac{1}{T} \sum_{s=1}^{T_E-T_B} \frac{1 - \gamma^s}{1 - \gamma} \alpha_2 Q_{T_B+s} \\ &= \frac{\alpha_2}{1 - \gamma} \left(\frac{T_E - T_B}{T} \text{plim} \frac{1}{T_E - T_B} \sum_{s=1}^{T_E-T_B} Q_{T_B+s} - \frac{1}{T} \sum_{s=1}^{T_E-T_B} \gamma^s Q_{T_B+s} \right) \\ &= \frac{\alpha_2}{1 - \gamma} (\lambda_E - \lambda_B) Q_C. \end{aligned} \tag{31}$$

The final result follows from (15), together with the fact that $\sum_{s=1}^{T_E-T_B} \gamma^s Q_{T_B+s} = O_P(1)$.

Using \widehat{O}_{1t} we have for the average estimated damage

$$\begin{aligned} \text{plim} \frac{1}{T} \widehat{CD}_1 &= \text{plim} \frac{1}{T_E - T_B} \sum_{s=1}^{T_E-T_B} \frac{1 - \hat{\gamma}^s}{1 - \hat{\gamma}} \hat{\alpha}_2 Q_{T_B+s} \\ &= \frac{\alpha_2}{1 - \gamma} (\lambda_E - \lambda_B) Q_C, \end{aligned}$$

which follows directly (31) and consistency of $\hat{\gamma}$. Using \widehat{O}_{2t} we have for the average estimated damage

$$\begin{aligned}\text{plim } \frac{1}{T} \widehat{CD}_2 &= \text{plim } \frac{1}{T} \sum_{t=T_B+1}^{T_E} (p_t - \widehat{bfp}_t) Q_t \\ &= \text{plim } \frac{1}{T} \sum_{t=T_B+1}^{T_E} (p_t - bfp_t) Q_t + \text{plim } \frac{1}{T} \sum_{t=T_B+1}^{T_E} (bfp_t - \widehat{bfp}_t) Q_t \\ &= \text{plim } \frac{1}{T} CD - \text{plim } \frac{1}{T} \sum_{s=1}^{T_E-T_B} (\widehat{bfp}_{T_B+s} - bfp_{T_B+s}) Q_{T_B+s}.\end{aligned}$$

We will analyze the limiting behavior of the second term in more detail under Assumption 2. Proposition 1 implies that

$$(\widehat{bfp}_{T_B+s} - bfp_{T_B+s}) Q_{T_B+s} = - \sum_{j=0}^{s-1} \gamma^j \varepsilon_{T_B+s-j} Q_{T_B+s} + O_P(T^{-1/2}).$$

Assumption 2 implies that

$$- \sum_{j=0}^{s-1} E [\gamma^j \varepsilon_{T_B+s-j} Q_{T_B+s}] \geq 0,$$

with equality if and only if Assumption 2 (ii) holds with equality. Applying a LLN we find

$$\text{plim } \frac{1}{T} \sum_{s=1}^{T_E-T_B} (\widehat{bfp}_{T_B+s} - bfp_{T_B+s}) Q_{T_B+s} \geq 0,$$

again with equality if and only if Assumption 2 (ii) holds with equality; this completes the proof. \square

Proof of Lemma 1. We define $T_b = \lambda_b T$ and $T_e = \lambda_e T$ with the break fractions λ_b and λ_e fixed numbers for all values of T . Regarding the various components of the OLS estimators we have:

$$\begin{aligned}\bar{d} &= \frac{1}{T} \sum_{t=1}^{T_b} d_t + \frac{1}{T} \sum_{t=T_b+1}^{T_e} d_t + \frac{1}{T} \sum_{t=T_e+1}^T d_t = \lambda_e - \lambda_b, \\ s_d^2 &= \bar{d}(1 - \bar{d}) = (\lambda_e - \lambda_b)(1 - \lambda_e + \lambda_b), \\ \bar{p} &= \frac{1}{T} \sum_{t=1}^{T_B} p_t + \frac{1}{T} \sum_{t=T_B+1}^{T_E} p_t + \frac{1}{T} \sum_{t=T_E+1}^T p_t \\ &= (\lambda_B + 1 - \lambda_E) \alpha_1 + (\lambda_E - \lambda_B) (\alpha_1 + \alpha_2) + \frac{1}{T} \sum_{t=1}^T \varepsilon_t, \\ s_{dp} &= \frac{1}{T} \sum_{t=1}^T d_t p_t - \bar{d} \bar{p}.\end{aligned}$$

We have four misdating scenarios specified in (20). For $T_b < T_B$, $T_e < T_E$ we have

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T d_t p_t &= \frac{1}{T} \sum_{t=1}^{T_b} d_t p_t + \frac{1}{T} \sum_{t=T_b+1}^{T_B} d_t p_t + \frac{1}{T} \sum_{t=T_B+1}^{T_e} d_t p_t + \frac{1}{T} \sum_{t=T_e+1}^{T_E} d_t p_t + \frac{1}{T} \sum_{t=T_E+1}^T d_t p_t \\ &= (\lambda_B - \lambda_b) \alpha_1 + (\lambda_e - \lambda_B) (\alpha_1 + \alpha_2) + \frac{1}{T} \sum_{t=1}^T d_t \varepsilon_t, \end{aligned}$$

provided that $T_B \leq T_e$; if $T_e < T_B$, then this simplifies to

$$\frac{1}{T} \sum_{t=1}^T d_t p_t = (\lambda_e - \lambda_b) \alpha_1 + \frac{1}{T} \sum_{t=1}^T d_t \varepsilon_t.$$

Analogously, for $T_b < T_B$, $T_e > T_E$ we find

$$\frac{1}{T} \sum_{t=1}^T d_t p_t = (\lambda_B - \lambda_b + \lambda_e - \lambda_E) \alpha_1 + (\lambda_E - \lambda_B) (\alpha_1 + \alpha_2) + \frac{1}{T} \sum_{t=1}^T d_t \varepsilon_t,$$

whereas for $T_b > T_B$, $T_e < T_E$, we have

$$\frac{1}{T} \sum_{t=1}^T d_t p_t = (\lambda_e - \lambda_b) (\alpha_1 + \alpha_2) + \frac{1}{T} \sum_{t=1}^T d_t \varepsilon_t.$$

Finally, for $T_b > T_B$, $T_e > T_E$ we have (provided that $T_E \geq T_b$)

$$\frac{1}{T} \sum_{t=1}^T d_t p_t = (\lambda_E - \lambda_b) (\alpha_1 + \alpha_2) + (\lambda_e - \lambda_E) \alpha_1 + \frac{1}{T} \sum_{t=1}^T d_t \varepsilon_t;$$

if $T_b > T_E$, this again simplifies to

$$\frac{1}{T} \sum_{t=1}^T d_t p_t = (\lambda_e - \lambda_b) \alpha_1 + \frac{1}{T} \sum_{t=1}^T d_t \varepsilon_t.$$

Taking probability limits we have:

$$\text{plim} \frac{1}{T} \sum_{t=1}^T \varepsilon_t = 0, \quad \text{plim} \frac{1}{T} \sum_{t=1}^T d_t \varepsilon_t = 0,$$

and

$$\text{plim} \bar{p} = \alpha_1 + (\lambda_E - \lambda_B) \alpha_2.$$

Regarding s_{dp} we have that

$$\text{plim} s_{dp} = \text{plim} \frac{1}{T} \sum_{t=1}^T d_t p_t - \text{plim} \bar{d} \text{plim} \bar{p},$$

and we find, using the notation $x^+ = \max\{x, 0\}$,

$$\begin{aligned} T_b < T_B, T_e < T_E : \quad \text{plim } s_{dp} &= ((\lambda_e - \lambda_B)^+ - (\lambda_e - \lambda_b)(\lambda_E - \lambda_B)) \alpha_2, \\ T_b < T_B, T_e > T_E : \quad \text{plim } s_{dp} &= (1 - \lambda_e + \lambda_b)(\lambda_E - \lambda_B) \alpha_2, \\ T_b > T_B, T_e < T_E : \quad \text{plim } s_{dp} &= (\lambda_e - \lambda_b)(1 - \lambda_E + \lambda_B) \alpha_2, \\ T_b > T_B, T_e > T_E : \quad \text{plim } s_{dp} &= ((\lambda_E - \lambda_b)^+ - (\lambda_e - \lambda_b)(\lambda_E - \lambda_B)) \alpha_2. \end{aligned}$$

Collecting terms we find for α_2 :

$$\begin{aligned} T_b < T_B, T_e < T_E : \quad \text{plim } \hat{\alpha}_2 &= \alpha_2 \left(\frac{(\lambda_e - \lambda_B)^+ - (\lambda_e - \lambda_b)(\lambda_E - \lambda_B)}{(\lambda_e - \lambda_b)(1 - \lambda_e + \lambda_b)} \right), \\ T_b < T_B, T_e > T_E : \quad \text{plim } \hat{\alpha}_2 &= \alpha_2 \left(\frac{\lambda_E - \lambda_B}{\lambda_e - \lambda_b} \right), \\ T_b > T_B, T_e < T_E : \quad \text{plim } \hat{\alpha}_2 &= \alpha_2 \left(\frac{1 - \lambda_E + \lambda_B}{1 - \lambda_e + \lambda_b} \right), \\ T_b > T_B, T_e > T_E : \quad \text{plim } \hat{\alpha}_2 &= \alpha_2 \left(\frac{(\lambda_E - \lambda_b)^+ - (\lambda_e - \lambda_b)(\lambda_E - \lambda_B)}{(\lambda_e - \lambda_b)(1 - \lambda_e + \lambda_b)} \right). \end{aligned}$$

It is easy to see that for the cases $T_b < T_B, T_e > T_E$ and $T_b > T_B, T_e < T_E$ we have that $\text{plim } \hat{\alpha}_2 < \alpha_2$. For the remaining cases, define:

$$a = (\lambda_e - \lambda_B)^+, \quad b = \lambda_e - \lambda_b, \quad c = \lambda_E - \lambda_B, \quad d = (\lambda_E - \lambda_b)^+.$$

For the case $T_b < T_B, T_e < T_E$, the attenuation bias $AB = \text{plim } \hat{\alpha}_2 / \alpha_2$ becomes

$$AB = \frac{a - bc}{b(1 - b)}.$$

We have $0 < b < 1$, hence $b(1 - b) > 0$. Furthermore, because $0 \leq a < c < 1$ we have that

$$AB = \frac{a - bc}{b(1 - b)} < \frac{a - ba}{b(1 - b)} = \frac{a}{b} < 1.$$

In a similar way one can show for the last case $T_b > T_B, T_e > T_E$ that $AB < 1$, hence $\text{plim } \hat{\alpha}_2 < \alpha_2$.

Regarding α_1 we have

$$\begin{aligned} \text{plim } \hat{\alpha}_1 &= \text{plim } \bar{p} - \text{plim } \hat{\alpha}_2 \text{plim } \bar{d} \\ &= \alpha_1 + \alpha_2 (\lambda_E - \lambda_B) - (\lambda_e - \lambda_b) \text{plim } \hat{\alpha}_2. \end{aligned}$$

For Case 1 we can write

$$\begin{aligned} \text{plim } \hat{\alpha}_1 &= \alpha_1 + \alpha_2 \left(c - \frac{a - bc}{b(1 - b)} b \right) \\ &> \alpha_1, \end{aligned}$$

because $c > a$. For Case 2 we simply have

$$\begin{aligned}\text{plim } \hat{\alpha}_1 &= \alpha_1 + \alpha_2 (\lambda_E - \lambda_B) - \alpha_2 \left(\frac{\lambda_E - \lambda_B}{\lambda_e - \lambda_b} \right) (\lambda_e - \lambda_b) \\ &= \alpha_1,\end{aligned}$$

hence OLS is consistent for α_1 . For Case 3 we have $c > b$ and

$$\begin{aligned}\text{plim } \hat{\alpha}_1 &= \alpha_1 + \alpha_2 \frac{c - b}{1 - b} \\ &> \alpha_1,\end{aligned}$$

while for Case 4 we have $d < c$ and

$$\begin{aligned}\text{plim } \hat{\alpha}_1 &= \alpha_1 + \alpha_2 \left(c - \frac{d - bc}{b(1 - b)} b \right) \\ &> \alpha_1,\end{aligned}$$

which completes the proof. □

Proof of Proposition 3.

- (i) This follows straightforwardly from Lemma 1 and the definition of bfp_t for the simplified DGP (16).
- (ii) From Lemma 1 we know that $0 < \text{plim } \hat{\alpha}_2 < \alpha_2$ always from which the result for \bar{O}_1 follows. Noting that

$$\begin{aligned}t \in \mathbb{T}_C : E[p_t] &= \alpha_1 + \alpha_2 \\ t \in \mathbb{T}_N : E[p_t] &= \alpha_1,\end{aligned}$$

it is easily seen that:

$$\begin{aligned}T_b < T_B, T_e < T_E : \quad &\text{plim } \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} p_t = \alpha_1 + \frac{(\lambda_e - \lambda_B)^+}{\lambda_e - \lambda_b} \alpha_2, \\ T_b < T_B, T_e > T_E : \quad &\text{plim } \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} p_t = \alpha_1 + \frac{\lambda_E - \lambda_B}{\lambda_e - \lambda_b} \alpha_2, \\ T_b > T_B, T_e < T_E : \quad &\text{plim } \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} p_t = \alpha_1 + \alpha_2, \\ T_b > T_B, T_e > T_E : \quad &\text{plim } \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} p_t = \alpha_1 + \frac{(\lambda_E - \lambda_b)^+}{\lambda_e - \lambda_b} \alpha_2.\end{aligned}$$

From Lemma 1 we know that $\text{plim } \hat{\alpha}_1 \geq \alpha_1$ and only when $T_b < T_B, T_e > T_E$ we have that $\text{plim } \hat{\alpha}_1 = \alpha_1$. Collecting terms we then conclude that

$$\begin{aligned} \text{plim } \bar{O}_2 &= \text{plim } \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} p_t - \text{plim } \hat{\alpha}_1 \\ &< \alpha_2. \end{aligned}$$

This completes the proof. □

Proof of Theorem 2. We have for the effective damage

$$\begin{aligned} \text{plim } \frac{1}{T} CD &= \text{plim } \frac{1}{T} \sum_{t=T_B+1}^{T_E} \alpha_2 Q_t \\ &= \alpha_2 (\lambda_E - \lambda_B) \text{plim } \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} Q_t \\ &= \alpha_2 (\lambda_E - \lambda_B) Q_C. \end{aligned}$$

We consider the four cases separately.

Case 1: $T_b < T_B, T_e < T_E$. We have

$$\begin{aligned} \text{plim } \frac{1}{T} \widehat{CD}_1 &= \text{plim } \frac{1}{T} \sum_{t=T_b+1}^{T_e} \hat{\alpha}_2 Q_t \\ &= \text{plim } \hat{\alpha}_2 \text{plim } \frac{1}{T} \sum_{t=T_b+1}^{T_e} Q_t \\ &= \alpha_2 \left(\frac{(\lambda_e - \lambda_B)^+ - (\lambda_e - \lambda_b) (\lambda_E - \lambda_B)}{(\lambda_e - \lambda_b) (1 - \lambda_e + \lambda_b)} \right) \\ &\quad \times ((\lambda_e - \lambda_b) Q_N + (\lambda_e - \lambda_B)^+ (Q_C - Q_N)). \end{aligned}$$

Comparing \widehat{CD}_1 with CD , when $Q_N \neq Q_C$ no systematic pattern emerges, i.e.

$$\text{plim } \frac{1}{T} \widehat{CD}_1 \not\geq \text{plim } \frac{1}{T} CD.$$

When $Q_N = Q_C$, however, some algebra shows that

$$\begin{aligned} \text{plim } \frac{1}{T} \widehat{CD}_1 &= \text{plim } \frac{1}{T} CD - \alpha_2 \frac{(\lambda_E - \lambda_B) - (\lambda_e - \lambda_B)^+}{1 - \lambda_e + \lambda_b} Q_C \\ &< \text{plim } \frac{1}{T} CD. \end{aligned}$$

Regarding \widehat{CD}_2 we have

$$\begin{aligned}\text{plim} \frac{1}{T} \widehat{CD}_2 &= \text{plim} \frac{1}{T} \sum_{t=T_b+1}^{T_e} (p_t - \hat{\alpha}_1) Q_t \\ &= \text{plim} \frac{1}{T} \sum_{t=T_b+1}^{T_e} p_t Q_t - \text{plim} \hat{\alpha}_1 \text{plim} \frac{1}{T} \sum_{t=T_b+1}^{T_e} Q_t.\end{aligned}$$

Assuming that $T_B \leq T_e$, Assumption 2 implies

$$\begin{aligned}\text{plim} \frac{1}{T} \sum_{t=T_b+1}^{T_e} p_t Q_t &= (\lambda_B - \lambda_b) \text{plim} \frac{1}{T_B - T_b} \sum_{t=T_b+1}^{T_B} p_t Q_t \\ &\quad + (\lambda_e - \lambda_B) \frac{1}{T_e - T_B} \sum_{t=T_B+1}^{T_e} p_t Q_t \\ &\leq (\lambda_B - \lambda_b) \text{plim} \frac{1}{T_B - T_b} \sum_{t=T_b+1}^{T_B} p_t \text{plim} \frac{1}{T_B - T_b} \sum_{t=T_b+1}^{T_B} Q_t \\ &\quad + (\lambda_e - \lambda_B) \frac{1}{T_e - T_B} \sum_{t=T_B+1}^{T_e} p_t \frac{1}{T_e - T_B} \sum_{t=T_B+1}^{T_e} Q_t \\ &= (\lambda_B - \lambda_b) \alpha_1 Q_N + (\lambda_e - \lambda_B) (\alpha_1 + \alpha_2) Q_C.\end{aligned}$$

Similarly, $\text{plim} T^{-1} \sum_{t=T_b+1}^{T_e} Q_t = (\lambda_e - \lambda_b) Q_N + (\lambda_e - \lambda_B) Q_C$, so that

$$\begin{aligned}\text{plim} \frac{1}{T} \widehat{CD}_2 &\leq (\lambda_B - \lambda_b) (\alpha_1 - \text{plim} \hat{\alpha}_1) Q_N + (\lambda_e - \lambda_B) (\alpha_1 + \alpha_2 - \text{plim} \hat{\alpha}_1) Q_C \\ &< (\lambda_E - \lambda_B) \alpha_2 Q_C,\end{aligned}$$

where the final inequality follows from $\text{plim} \hat{\alpha}_1 > \alpha_1$ and $\lambda_e < \lambda_E$. In the special case $T_e < T_B$, the result simplifies to

$$\begin{aligned}\text{plim} \frac{1}{T} \widehat{CD}_2 &\leq (\lambda_e - \lambda_b) (\alpha_1 - \text{plim} \hat{\alpha}_1) Q_N \\ &< (\lambda_E - \lambda_B) \alpha_2 Q_C,\end{aligned}$$

simply because the probability limit of $T^{-1} \widehat{CD}_2$ is negative in this case (and $(\lambda_E - \lambda_B) \alpha_2 Q_C$ is positive).

Case 2: $T_b < T_B, T_e > T_E$. We have

$$\begin{aligned}\text{plim} \frac{1}{T} \widehat{CD}_1 &= \text{plim} \hat{\alpha}_2 \text{plim} \frac{1}{T} \sum_{t=T_b+1}^{T_e} Q_t \\ &= \text{plim} \hat{\alpha}_2 ((\lambda_e - \lambda_b) Q_N + (\lambda_E - \lambda_B) (Q_C - Q_N)) \\ &\geq \alpha_2 (\lambda_E - \lambda_B) Q_C,\end{aligned}$$

because $Q_N \geq Q_C$ and where we substituted $\text{plim } \hat{\alpha}_2 = \alpha_2(\lambda_E - \lambda_B)/(\lambda_e - \lambda_b)$; and, analogously to the proof for Case 1,

$$\begin{aligned} \text{plim } \frac{1}{T} \widehat{CD}_2 &\leq (\lambda_B - \lambda_b + \lambda_e - \lambda_E) (\alpha_1 - \text{plim } \hat{\alpha}_1) Q_N + (\lambda_E - \lambda_B) (\alpha_1 + \alpha_2 - \text{plim } \hat{\alpha}_1) Q_C \\ &= \alpha_2 (\lambda_E - \lambda_B) Q_C, \end{aligned}$$

because $\text{plim } \hat{\alpha}_1 = \alpha_1$.

Case 3: $T_b > T_B, T_e < T_E$. We have

$$\begin{aligned} \text{plim } \frac{1}{T} \widehat{CD}_1 &= \text{plim } \hat{\alpha}_2 \text{plim } \frac{1}{T} \sum_{t=T_b+1}^{T_e} Q_t \\ &= \text{plim } \hat{\alpha}_2 (\lambda_e - \lambda_b) Q_C \\ &< \alpha_2 (\lambda_E - \lambda_B) Q_C, \end{aligned}$$

because $\text{plim } \hat{\alpha}_2 < \alpha_2$ and $\lambda_e - \lambda_b < \lambda_E - \lambda_B$; and

$$\begin{aligned} \text{plim } \frac{1}{T} \widehat{CD}_2 &\leq (\lambda_e - \lambda_b) (\alpha_1 + \alpha_2 - \text{plim } \hat{\alpha}_1) Q_C \\ &< \alpha_2 (\lambda_E - \lambda_B) Q_C, \end{aligned}$$

because $\text{plim } \hat{\alpha}_1 > \alpha_1$ and $\lambda_e - \lambda_b < \lambda_E - \lambda_B$.

Case 4: $T_b > T_B, T_e > T_E$. We have

$$\begin{aligned} \text{plim } \frac{1}{T} \widehat{CD}_1 &= \alpha_2 \left(\frac{(\lambda_E - \lambda_b)^+ - (\lambda_e - \lambda_b) (\lambda_E - \lambda_B)}{(\lambda_e - \lambda_b) (1 - \lambda_e + \lambda_b)} \right) \\ &\quad \times ((\lambda_e - \lambda_b) Q_N + (\lambda_E - \lambda_b)^+ (Q_C - Q_N)). \end{aligned}$$

Comparing \widehat{CD}_1 with CD , when $Q_N \neq Q_C$ no systematic pattern emerges, i.e.

$$\text{plim } \frac{1}{T} \widehat{CD}_1 \gtrless \text{plim } \frac{1}{T} CD.$$

When $Q_N = Q_C$, however, some algebra shows that

$$\begin{aligned} &\alpha_2 Q_C \left(\frac{(\lambda_E - \lambda_b)^+ - (\lambda_E - \lambda_B)}{1 - \lambda_e + \lambda_b} \right) \\ \text{plim } \frac{1}{T} \widehat{CD}_1 &= \text{plim } \frac{1}{T} CD - \alpha_2 \frac{(\lambda_E - \lambda_B) - (\lambda_E - \lambda_b)^+}{1 - \lambda_e + \lambda_b} Q_C \\ &< \text{plim } \frac{1}{T} CD. \end{aligned}$$

Regarding \widehat{CD}_2 we have (provided that $T_E \geq T_b$)

$$\begin{aligned} \text{plim } \frac{1}{T} \widehat{CD}_2 &\leq (\lambda_E - \lambda_b) (\alpha_1 + \alpha_2 - \text{plim } \hat{\alpha}_1) Q_C + (\lambda_e - \lambda_E) (\alpha_1 - \text{plim } \hat{\alpha}_1) Q_N \\ &< (\lambda_E - \lambda_B) \alpha_2 Q_C, \end{aligned}$$

for because $\text{plim } \hat{\alpha}_1 > \alpha_1$ and $\lambda_b > \lambda_B$. For the special case $T_E < T_b$, the probability limit reduces to

$$(\lambda_e - \lambda_b)(\alpha_1 - \text{plim } \hat{\alpha}_1)Q_N < 0 < (\lambda_E - \lambda_B)\alpha_2Q_C.$$

This completes the proof. \square

Proof of Lemma 2. It is helpful for further calculations on the OLS inconsistency to invoke the result that without loss of generalization we can assume that x_t has mean zero, hence

$$m_x = \text{plim } \frac{1}{T} \sum_{t=1}^T x_t = 0.$$

Moreover, we can redefine the cartel dummy variable such that actually the measurement error $v_t = d_t - D_t$ has mean zero. The reason is that we have a constant term in the model, so all variables can be taken in deviation from their sample average, affecting only the definition and interpretation of the intercept. We redefine D_t and d_t as dummy variables in deviation from their sample mean:

$$\begin{aligned} D_t &= -\frac{T_E - T_B}{T} = \lambda_B - \lambda_E, & t \leq T_B, \\ D_t &= \frac{T - T_E + T_B}{T} = 1 - \lambda_E + \lambda_B, & T_B < t \leq T_E, \\ D_t &= -\frac{T_E - T_B}{T} = \lambda_B - \lambda_E, & t > T_E, \end{aligned}$$

and:

$$\begin{aligned} d_t &= -\frac{T_e - T_b}{T} = \lambda_b - \lambda_e, & t \leq T_b, \\ d_t &= \frac{T - T_e + T_b}{T} = 1 - \lambda_e + \lambda_b, & T_b < t \leq T_e, \\ d_t &= -\frac{T_e - T_b}{T} = \lambda_b - \lambda_e, & t > T_e. \end{aligned}$$

The result is that, irrespective of whether we date T_b and T_e too early or too late, the average measurement error is zero, i.e.

$$\frac{1}{T} \sum_{t=1}^T v_t = 0.$$

If we let D_t^0 and d_t^0 denote the original 0–1 dummy variables, such that $D_t = D_t^0 - (\lambda_E - \lambda_B)$ and $d_t = d_t^0 - (\lambda_e - \lambda_b)$, then

$$\begin{aligned} p_t &= \alpha_1 + \alpha_2 D_t^0 + \beta' x_t + \gamma p_{t-1} + \varepsilon_t \\ &= \alpha + \alpha_2 D_t + \beta' x_t + \gamma p_{t-1} + \varepsilon_t, \end{aligned}$$

so that the new intercept becomes

$$\alpha = \alpha_1 + (\lambda_E - \lambda_B) \alpha_2.$$

Analogously, the estimated model becomes

$$\begin{aligned} \hat{p}_t &= \hat{\alpha}_1 + \hat{\alpha}_2 d_t^0 + \hat{\beta}' x_t + \hat{\gamma} p_{t-1} \\ &= \hat{\alpha} + \hat{\alpha}_2 d_t + \hat{\beta}' x_t + \hat{\gamma} p_{t-1}, \end{aligned}$$

with

$$\hat{\alpha} = \hat{\alpha}_1 + (\lambda_e - \lambda_b) \hat{\alpha}_2.$$

Stacking the observations ($t = 1, \dots, T$), we write the regression model to be estimated as

$$y = Z\theta + u,$$

where $y = (p_1, \dots, p_T)'$ and $u = (u_1, \dots, u_T)'$. Furthermore, $Z = (z_1, \dots, z_T)'$ with $z_t = (p_{t-1}, x_t, 1, d_t)'$ and $\theta = (\gamma, \beta, \alpha, \alpha_2)'$. The OLS estimator of the full parameter vector θ is equal to

$$\hat{\theta} = (Z'Z)^{-1}Z'y.$$

Taking the probability limit we have

$$\begin{aligned} \text{plim } \hat{\theta} &= \theta + \left(\text{plim } \frac{1}{T} Z'Z \right)^{-1} \text{plim } \frac{1}{T} Z'u \\ &= \theta + \Sigma_{ZZ}^{-1} \Sigma_{Zu}. \end{aligned} \tag{32}$$

The vector $\Sigma_{ZZ}^{-1} \Sigma_{Zu}$ is the OLS inconsistency.

Regarding Σ_{Zu} in (32) we have:

$$\begin{aligned} m_{xu} &= \text{plim } \frac{1}{T} \sum_{t=1}^T x_t u_t = 0, \\ m_u &= \text{plim } \frac{1}{T} \sum_{t=1}^T u_t = 0, \end{aligned}$$

where $u_t = \varepsilon_t - \alpha_2 v_t$. Also we have

$$E[d_t u_t] = -\alpha_2 E[d_t v_t] \neq 0,$$

hence $m_{du} \neq 0$. Also there is a non-zero correlation between p_{t-1} and u_t . Exploiting the stationarity assumption by repeated substitution we can write

$$p_t = \beta \sum_{s=0}^{\infty} \gamma^s x_{t-s} + \frac{\alpha}{1-\gamma} + \alpha_2 \sum_{s=0}^{\infty} \gamma^s D_{t-s} + \sum_{s=0}^{\infty} \gamma^s \varepsilon_{t-s},$$

so we have (exploiting the fact that ε_t has no autocorrelation, and zero correlation with lagged x_{t-s-1} by Assumption 1) that

$$\begin{aligned} E[p_{t-1}u_t] &= -\alpha_2^2 E\left[v_t \sum_{s=0}^{\infty} \gamma^s D_{t-1-s}\right] \\ &= -\alpha_2^2 (v_t D_{t-1} + \gamma v_t D_{t-2} + \gamma^2 v_t D_{t-3} + \dots) \\ &\neq 0, \end{aligned}$$

so $m_{p-1u} \neq 0$. Under the assumptions, we have now:

$$\Sigma_{ZZ} = \begin{pmatrix} m_{pp} & m_{xp-1} & m_p & m_{dp-1} \\ m_{xp-1} & m_{xx} & 0 & 0 \\ m_p & 0 & 1 & 0 \\ m_{dp-1} & 0 & 0 & m_{dd} \end{pmatrix}, \quad \Sigma_{Zu} = \begin{bmatrix} m_{p-1u} \\ 0 \\ 0 \\ m_{du} \end{bmatrix}.$$

After some algebra we find for the inconsistency

$$\Sigma_{ZZ}^{-1} \Sigma_{Zu} = \frac{1}{\det(\Sigma_{ZZ})} \begin{bmatrix} m_{xx} (m_{dd} m_{p-1u} - m_{du} m_{dp-1}) \\ -m_{xp-1} (m_{dd} m_{p-1u} - m_{du} m_{dp-1}) \\ -m_p m_{xx} (m_{dd} m_{p-1u} - m_{du} m_{dp-1}) \\ -m_{dp-1} m_{xx} m_{p-1u} - m_{du} (\sigma_{xp-1}^2 - \sigma_x^2 \sigma_p^2) \end{bmatrix}, \quad (33)$$

where $\det(\Sigma_{ZZ}) > 0$ and we define $\sigma_p^2 = m_{pp} - m_p^2$. Furthermore, because we assume $m_x = 0$, we have $m_{xx} = \sigma_x^2$ and $m_{xp-1} = \sigma_{xp-1}$.

We now have to evaluate all separate terms in (33). We always have

$$\begin{aligned} m_{dd} &= \text{plim} \frac{1}{T} \sum d_t^2 \\ &= (1 - \lambda_e + \lambda_b) (\lambda_e - \lambda_b). \end{aligned}$$

Furthermore, to evaluate m_{p-1u} we note that

$$\begin{aligned}
m_{p-1u} &= \text{plim} \frac{1}{T} \sum_{t=1}^T p_{t-1} u_t \\
&= -\alpha_2^2 \text{plim} \frac{1}{T} \sum_{t=1}^T (v_t D_{t-1} + \gamma v_t D_{t-2} + \gamma^2 v_t D_{t-3} + \dots) \\
&= -\alpha_2^2 \text{plim} \frac{1}{T} \sum_{t=1}^T \sum_{s=0}^{\infty} \gamma^s v_t D_{t-1-s} \\
&= -\alpha_2^2 \text{plim} \frac{1}{T} \sum_{t=1}^T \sum_{s=0}^{\infty} \gamma^s v_t (D_t + D_{t-1-s} - D_t) \\
&= -\frac{\alpha_2^2}{1-\gamma} \text{plim} \frac{1}{T} \sum_{t=1}^T v_t D_t + \alpha_2^2 \text{plim} \frac{1}{T} \sum_{t=1}^T \sum_{s=0}^{\infty} \gamma^s v_t (D_t - D_{t-1-s}) \\
&= -\frac{\alpha_2^2}{1-\gamma} \text{plim} \frac{1}{T} \sum_{t=1}^T v_t D_t.
\end{aligned}$$

The final equality holds because $v_t (D_t - D_{t-1-s}) \propto \frac{s+1}{T}$, because $D_t - D_{t-1-s}$ has nonzero values in (two times) $\frac{s+1}{T}$ observations only. Therefore,

$$\begin{aligned}
\sum_{s=0}^{\infty} \gamma^s v_t (D_t - D_{t-1-s}) &\propto \sum_{s=0}^{\infty} \gamma^s \frac{s+1}{T} \\
&= \frac{1}{T} \sum_{s=0}^{\infty} (s+1) \gamma^s \\
&= \frac{1}{T(1-\gamma)^2},
\end{aligned}$$

which is of order $O(T^{-1})$ only.

For the same reason we have

$$\begin{aligned}
m_{dp-1} &= \text{plim} \frac{1}{T} \alpha_2 \sum_{t=1}^T \sum_{s=0}^{\infty} \gamma^s d_t D_{t-1-s} \\
&= \alpha_2 \text{plim} \frac{1}{T} \sum_{t=1}^T \sum_{s=0}^{\infty} \gamma^s d_t (D_t - (D_t - D_{t-1-s})) \\
&= \frac{\alpha_2}{1-\gamma} \text{plim} \frac{1}{T} \sum_{t=1}^T d_t D_t.
\end{aligned}$$

The precise magnitude of the separate terms will depend on the type of break misdating. We will provide detailed derivations for Cases 1 and 2 only. In Case 1 ($T_b < T_B < T_e < T_E$)

we have for the measurement error:

$$v_t = \begin{cases} \lambda_b - \lambda_e - \lambda_B + \lambda_E, & t \leq T_b, \\ 1 - \lambda_e + \lambda_b - \lambda_B + \lambda_E, & T_b < t \leq T_B, \\ \lambda_b - \lambda_e - \lambda_B + \lambda_E, & T_B < t \leq T_e, \\ \lambda_b - \lambda_e - 1 - \lambda_B + \lambda_E, & T_e < t \leq T_E, \\ \lambda_b - \lambda_e - \lambda_B + \lambda_E, & t > T_E. \end{cases}$$

We then have

$$\begin{aligned} m_{du} &= -\alpha_2 \text{plim} \frac{1}{T} \sum_{t=1}^T d_t v_t \\ &= -\alpha_2 [\lambda_b (\lambda_b - \lambda_e) (\lambda_b - \lambda_e - \lambda_B + \lambda_E) \\ &\quad + (\lambda_B - \lambda_b) (1 - \lambda_e + \lambda_b) (1 - \lambda_e + \lambda_b - \lambda_B + \lambda_E) \\ &\quad + (\lambda_e - \lambda_B) (1 - \lambda_e + \lambda_b) (-\lambda_e + \lambda_b - \lambda_B + \lambda_E) \\ &\quad + (\lambda_E - \lambda_e) (\lambda_b - \lambda_e) (-1 - \lambda_e + \lambda_b - \lambda_B + \lambda_E) \\ &\quad + (1 - \lambda_E) (\lambda_b - \lambda_e) (\lambda_b - \lambda_e - \lambda_B + \lambda_E)] \\ &= -\alpha_2 [(\lambda_B - \lambda_b) (1 - \lambda_e + \lambda_b) + (\lambda_e - \lambda_b) (\lambda_E - \lambda_e)]. \end{aligned}$$

Furthermore, we have that

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T d_t D_t &= \lambda_b (\lambda_b - \lambda_e) (\lambda_B - \lambda_E) \\ &\quad + (\lambda_B - \lambda_b) (1 - \lambda_e + \lambda_b) (\lambda_B - \lambda_E) \\ &\quad + (\lambda_e - \lambda_B) (1 - \lambda_e + \lambda_b) (1 - \lambda_E + \lambda_B) \\ &\quad + (\lambda_E - \lambda_e) (\lambda_b - \lambda_e) (1 - \lambda_E + \lambda_B) \\ &\quad + (1 - \lambda_E) (\lambda_b - \lambda_e) (\lambda_B - \lambda_E) \\ &= \lambda_e - \lambda_B + (\lambda_e - \lambda_b) (\lambda_B - \lambda_E), \end{aligned}$$

hence we find that

$$m_{dp-1} = \frac{\alpha_2}{1 - \gamma} (\lambda_e - \lambda_B + (\lambda_e - \lambda_b) (\lambda_B - \lambda_E)).$$

Together with

$$\frac{1}{T} \sum_{t=1}^T D_t^2 = (1 - \lambda_E + \lambda_B) (\lambda_E - \lambda_B),$$

we find

$$\begin{aligned} \frac{1}{T} \sum_{t=1}^T v_t D_t &= \frac{1}{T} \sum_{t=1}^T d_t D_t - \frac{1}{T} \sum_{t=1}^T D_t^2, \\ &= \lambda_e - \lambda_B + (1 - \lambda_E + \lambda_B + \lambda_e - \lambda_b) (\lambda_B - \lambda_E), \end{aligned}$$

and we therefore have

$$m_{p-1u} = -\frac{\alpha_2^2}{1-\gamma} [\lambda_e - \lambda_B + (1 - \lambda_E + \lambda_B + \lambda_e - \lambda_b)(\lambda_B - \lambda_E)].$$

Inspecting signs, we obviously have $m_{dd} > 0$. Furthermore, assuming the cartel effect $\alpha_2 > 0$ and given $T_b < T_B < T_e < T_E$ we find that $m_{du} < 0$ and $m_{dp-1} > 0$ because

$$\begin{aligned} \lambda_e - \lambda_B + (\lambda_e - \lambda_b)(\lambda_B - \lambda_E) &< \lambda_E - \lambda_B + (\lambda_e - \lambda_b)(\lambda_B - \lambda_E) \\ &= (\lambda_E - \lambda_B)(1 + \lambda_b - \lambda_e) \\ &> 0. \end{aligned}$$

Also we find $m_{p-1u} > 0$ because

$$\begin{aligned} &(\lambda_E - \lambda_B)(1 - \lambda_E + \lambda_B + \lambda_e - \lambda_b) - (\lambda_e - \lambda_B) \\ &> (\lambda_E - \lambda_B)(1 - \lambda_E + \lambda_B + \lambda_e - \lambda_B) - (\lambda_e - \lambda_B) \\ &= (1 - \lambda_E + \lambda_B)(\lambda_E - \lambda_e) \\ &> 0, \end{aligned}$$

where the second line follows from the fact that $\lambda_B > \lambda_b$. Collecting terms we therefore find for Case 1 that

$$m_{dd}m_{p-1u} - m_{du}m_{dp-1} > 0.$$

Hence, we can write for the direction of the first three elements in the OLS inconsistency (33):

$$\begin{aligned} \text{plim } \hat{\gamma} &> \gamma, \\ \text{plim } \hat{\beta} &\leq \beta, \quad \text{if } m_{xp-1} \geq 0, \\ \text{plim } \hat{\alpha} &< \alpha, \end{aligned}$$

where the last inequality holds as prices are positive, hence $m_{p-1} > 0$. Finally, the last element in the inconsistency is

$$\begin{aligned} \text{plim } (\hat{\alpha}_2 - \alpha_2) &= \frac{-m_{dp-1}m_{xx}m_{p-1u} - m_{du}(\sigma_{xp-1}^2 - \sigma_x^2\sigma_p^2)}{\det(\Sigma_{ZZ})} \\ &= -\sigma_x^2 \frac{m_{dp-1}m_{p-1u} - m_{du}\sigma_p^2(1 - \rho_{xp-1}^2)}{\det(\Sigma_{ZZ})} \\ &< 0, \end{aligned}$$

because $m_{dp-1} > 0$, $m_{p-1u} > 0$ and $m_{du} < 0$.

We find exactly the same qualitative results for the Cases 2, 3 and 4. Because Case 2 will turn out to be most important we briefly state the results for this case. In Case 2, i.e. $T_b < T_B, T_e > T_E$, we have for the measurement error:

$$v_t = \begin{cases} \lambda_b - \lambda_e - \lambda_B + \lambda_E, & t \leq T_b, \\ 1 - \lambda_e + \lambda_b - \lambda_B + \lambda_E, & T_b < t \leq T_B, \\ \lambda_b - \lambda_e - \lambda_B + \lambda_E, & T_B < t \leq T_E, \\ 1 - \lambda_e + \lambda_b - \lambda_B + \lambda_E, & T_E < t \leq T_e, \\ \lambda_b - \lambda_e - \lambda_B + \lambda_E, & t > T_e. \end{cases}$$

We then have

$$\begin{aligned} m_{du} &= -\alpha_2 (1 - \lambda_e + \lambda_b) (\lambda_B - \lambda_E + \lambda_e - \lambda_b), \\ m_{dp-1} &= \frac{\alpha_2}{1 - \gamma} (\lambda_E - \lambda_B) (1 - \lambda_e + \lambda_b), \\ m_{p-1u} &= -\frac{\alpha_2^2}{1 - \gamma} (\lambda_E - \lambda_B) (\lambda_b - \lambda_e + \lambda_E - \lambda_B). \end{aligned}$$

Inspecting signs, we find that $m_{du} < 0$ as

$$\begin{aligned} (1 - \lambda_e + \lambda_b) (\lambda_B - \lambda_E + \lambda_e - \lambda_b) &= (1 - (\lambda_e - \lambda_b)) (\lambda_e - \lambda_b - (\lambda_E - \lambda_B)) \\ &> 0, \end{aligned}$$

because in Case 2 $\lambda_e - \lambda_b > \lambda_E - \lambda_B$. Furthermore, it is obvious that $m_{dp-1} > 0$ and $m_{p-1u} > 0$. Collecting terms we therefore find the same qualitative results compared with Case 1.

For the original intercept α_1 , we obtain the following. Defining

$$c = \frac{m_{dd}m_{p-1u} - m_{du}m_{dp-1}}{\det(\Sigma_{ZZ})},$$

we can write

$$\begin{aligned} \text{plim} \left(\frac{\hat{\alpha}}{1 - \hat{\gamma}} - \frac{\alpha}{1 - \gamma} \right) &= \frac{\text{plim}(\hat{\alpha} - \alpha) + \frac{\alpha}{1 - \gamma} \text{plim}(\hat{\gamma} - \gamma)}{1 - \gamma - \text{plim}(\hat{\gamma} - \gamma)} \\ &= \frac{-m_p m_{xx} c + \frac{\alpha}{1 - \gamma} m_{xx} c}{1 - \gamma - m_{xx} c} \\ &= \left(\frac{m_{xx} c}{1 - \gamma - m_{xx} c} \right) \left(\frac{\alpha}{1 - \gamma} - m_p \right) \\ &= 0. \end{aligned}$$

It can also be shown that:

$$(\lambda_E - \lambda_B) \frac{\alpha_2}{1 - \gamma} - (\lambda_e - \lambda_b) \text{plim} \frac{\hat{\alpha}_2}{1 - \hat{\gamma}} = \begin{cases} 0, & T_b < T_B, T_e > T_E, \\ > 0, & \text{otherwise.} \end{cases} \quad (34)$$

Therefore, given that $\text{plim } \hat{\gamma} > \gamma$ it is easily seen that for Case 2

$$\begin{aligned}\text{plim } \hat{\alpha}_1 &= \text{plim } \hat{\alpha} - (\lambda_e - \lambda_b) \text{plim } \hat{\alpha}_2 \\ &= \alpha_1 \frac{\text{plim } (1 - \hat{\gamma})}{1 - \gamma} \\ &< \alpha_1.\end{aligned}$$

Numerical simulations confirm that it holds for the other cases also. \square

Proof of Proposition 4.

(i) Recall that

$$\begin{aligned}t \leq T_b : \widehat{bfp}_t &= p_t, \\ t > T_b : \widehat{bfp}_t &= \hat{\gamma} \widehat{bfp}_{t-1} + \hat{\beta} x_t + \hat{\alpha}_1,\end{aligned}$$

and

$$bfp_t = \gamma bfp_{t-1} + \beta x_t + \alpha_1 + \varepsilon_t,$$

where in terms of the parameter vector $\theta = (\gamma, \beta, \alpha, \alpha_1)'$, we have $\hat{\alpha}_1 = \hat{\alpha} - (\lambda_e - \lambda_b) \hat{\alpha}_2$ and $\alpha_1 = \alpha - (\lambda_E - \lambda_B) \alpha_2$. We have for the average estimated but-for price

$$\begin{aligned}\text{plim } \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} \widehat{bfp}_t &= \text{plim } \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} bfp_t + \text{plim } \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} (\widehat{bfp}_t - bfp_t) \\ &= \text{plim } \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} bfp_t + \text{plim } \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} (\widehat{bfp}_t - bfp_t).\end{aligned}$$

We will analyze the limiting behavior of the second term in more detail. In period $T_b + 1$ we can write for the prediction error

$$\begin{aligned}v_{T_b+1} &= \widehat{bfp}_{T_b+1} - bfp_{T_b+1} \\ &= (\hat{\gamma} - \gamma) bfp_{T_b} + \hat{\gamma} (p_{T_b} - bfp_{T_b}) + (\hat{\beta} - \beta) x_{T_b+1} + \hat{\alpha} - \alpha - \varepsilon_{T_b+1}.\end{aligned}$$

Note that standard asymptotic theory for method of moments estimators gives the following large sample distribution of the OLS estimator

$$\sqrt{T} \left(\hat{\theta} - \theta - \theta^* \right) \xrightarrow{d} \mathcal{N}(0, V),$$

where $\theta^* = \Sigma_{ZZ}^{-1} \Sigma_{Zu}$ is the inconsistency. Therefore, we can write $\hat{\theta} - \theta = \theta^* + O_P(T^{-1/2})$. The implied inconsistency in α_1 is

$$\alpha_1^* = \alpha^* - (\lambda_e - \lambda_b) \alpha_2^* + (\lambda_E - \lambda_B - (\lambda_e - \lambda_b)) \alpha_2.$$

Also note that we have:

$$bfp_{T_b} = \begin{cases} p_{T_b}, & T_b \leq T_B, \\ < p_{T_b}, & T_b > T_B, \end{cases}$$

so that

$$\hat{\gamma}(p_{T_b} - bfp_{T_b}) = \begin{cases} 0, & T_b \leq T_B, \\ (\gamma + \gamma^*)(p_{T_b} - bfp_{T_b}) + O_P(T^{-1/2}), & T_b > T_B. \end{cases}$$

Therefore we can write:

$$v_{T_b+1} = \begin{cases} \gamma^* bfp_{T_b} + \beta^* x_{T_b+1} + \alpha_1^* - \varepsilon_{T_b+1} + O_P(T^{-1/2}), & T_b \leq T_B, \\ \gamma^* bfp_{T_b} + \beta^* x_{T_b+1} + \alpha_1^* - \varepsilon_{T_b+1} + O_P(1) & T_b > T_B. \end{cases}$$

In period $T_b + 2$ we have for the difference in estimated and true but-for prices

$$\begin{aligned} \widehat{bfp}_{T_b+2} - bfp_{T_b+2} &= \hat{\gamma} \widehat{bfp}_{T_b+1} - \gamma bfp_{T_b+1} + (\hat{\beta} - \beta)x_{T_b+2} + \hat{\alpha}_1 - \alpha_1 - \varepsilon_{T_b+2} \\ &= \hat{\gamma} v_{T_b+1} + (\hat{\gamma} - \gamma) bfp_{T_b+1} + (\hat{\beta} - \beta)x_{T_b+2} + \hat{\alpha}_1 - \alpha_1 - \varepsilon_{T_b+2} \\ &= \hat{\gamma} v_{T_b+1} + v_{T_b+2}. \end{aligned}$$

Regarding the prediction error v_{T_b+2} we find

$$v_{T_b+2} = \gamma^* bfp_{T_b+1} + \beta^* x_{T_b+2} + \alpha_1^* - \varepsilon_{T_b+2} + O_P(T^{-1/2}),$$

irrespective of the precise timing of T_b and T_B . In general we find

$$\begin{aligned} \widehat{bfp}_{T_b+s} - bfp_{T_b+s} &= \hat{\gamma}^{s-1} v_{T_b+1} + \hat{\gamma}^{s-2} v_{T_b+2} + \dots + \hat{\gamma} v_{T_b+s-1} + v_{T_b+s} \\ &= \sum_{j=0}^{s-1} \hat{\gamma}^j v_{T_b+s-j} \\ &= \sum_{j=0}^{s-1} (\gamma + \gamma^*)^j v_{T_b+s-j} + O_P(T^{-1/2}), \end{aligned}$$

where

$$v_{T_b+s-j} = \gamma^* bfp_{T_b+s-j-1} + \beta^* x_{T_b+s-j} + \alpha_1^* - \varepsilon_{T_b+s-j} + O_P(T^{-1/2}), \quad j = 0, \dots, s-2.$$

By repeated substitution we can write

$$bfp_t = \frac{\alpha_1}{1 - \gamma} + \beta \sum_{i=0}^{\infty} \gamma^i x_{t-i} + \sum_{i=0}^{\infty} \gamma^i \varepsilon_{t-i},$$

hence

$$E[bfp_t] = \frac{\alpha_1}{1 - \gamma},$$

because $E[x_t] = 0$ and $E[\varepsilon_t] = 0$. Defining

$$c = \frac{m_{dd}m_{p-1u} - m_{du}m_{dp-1}}{\det(\Sigma_{ZZ})},$$

we can write

$$\begin{aligned} \text{plim} \left(\frac{\hat{\alpha}}{1 - \hat{\gamma}} - \frac{\alpha}{1 - \gamma} \right) &= \frac{\text{plim}(\hat{\alpha} - \alpha) + \alpha \text{plim}(\hat{\gamma} - \gamma)/(1 - \gamma)}{1 - \gamma - \text{plim}(\hat{\gamma} - \gamma)} \\ &= \frac{-m_p m_{xx} c + m_{xx} c \alpha / (1 - \gamma)}{1 - \gamma - m_{xx} c} \\ &= 0, \end{aligned}$$

because $m_p = \alpha/(1 - \gamma)$.

Using $\gamma^* = m_{xx} c$ and $\alpha^* = -m_p m_{xx} c$ once more we get

$$\begin{aligned} E[v_{T_b+s-j}] &= E[\gamma^* bfp_{T_b+s-j-1} + \beta^* x_{T_b+s-j} + \alpha_1^* - \varepsilon_{T_b+s-j}] + O(T^{-1}) \\ &= m_{xx} c \left(\frac{\alpha}{1 - \gamma} - (\lambda_E - \lambda_B) \frac{\alpha_2}{1 - \gamma} \right) - m_p m_{xx} c \\ &\quad - ((\lambda_e - \lambda_b) \text{plim} \hat{\alpha}_2 - (\lambda_E - \lambda_B) \alpha_2) + O(T^{-1}) \\ &= -\text{plim}(\hat{\gamma} - \gamma) (\lambda_E - \lambda_B) \frac{\alpha_2}{1 - \gamma} - (\lambda_e - \lambda_b) \text{plim} \hat{\alpha}_2 \\ &\quad + (\lambda_E - \lambda_B) \alpha_2 + O(T^{-1}) \\ &= (\lambda_E - \lambda_B) \frac{\alpha_2}{1 - \gamma} (1 - \text{plim} \hat{\gamma}) - (\lambda_e - \lambda_b) \text{plim} \hat{\alpha}_2. \end{aligned}$$

From (34) we see that:

$$\text{plim} \hat{\alpha}_2 = \begin{cases} \frac{\lambda_E - \lambda_B}{\lambda_e - \lambda_b} \frac{\alpha_2}{1 - \gamma} (1 - \text{plim} \hat{\gamma}), & T_b < T_B, T_e > T_E, \\ < \frac{\lambda_E - \lambda_B}{\lambda_e - \lambda_b} \frac{\alpha_2}{1 - \gamma} (1 - \text{plim} \hat{\gamma}), & \text{otherwise.} \end{cases} \quad (35)$$

Using this we find:

$$E[v_{T_b+s-j}] = \begin{cases} O(T^{-1}), & T_b < T_B, T_e > T_E, \\ O(1) > 0, & \text{otherwise.} \end{cases}$$

The difference in estimated and true but-for price is

$$\widehat{bfp}_{T_b+s} - bfp_{T_b+s} = \sum_{j=0}^{s-1} (\gamma + \gamma^*)^j v_{T_b+s-j} + O_P(T^{-1/2}),$$

with:

$$E \left[\sum_{j=0}^{s-1} (\gamma + \gamma^*)^j v_{T_b+s-j} \right] = \begin{cases} O(T^{-1}), & T_b < T_B, T_e > T_E, \\ O(1) > 0, & \text{otherwise.} \end{cases}$$

Hence:

$$\begin{aligned} & \text{plim} \frac{1}{T_e - T_b} \sum_{s=1}^{T_e - T_b} \left(\widehat{bfp}_{T_b+s} - bfp_{T_b+s} \right) \\ &= \text{plim} \frac{1}{T_e - T_b} \sum_{s=1}^{T_e - T_b} \sum_{j=0}^{s-1} (\gamma + \gamma^*)^j v_{T_b+s-j} = \begin{cases} 0, & T_b < T_B, T_e > T_E, \\ > 0, & \text{otherwise,} \end{cases} \end{aligned}$$

which completes the proof of (i).

(ii) Noting that $bfp_{T_B} = p_{T_B}$ we have for the overcharge

$$O_{T_B+t} = (p_{T_B+t} - bfp_{T_B+t}) D_t = \frac{1 - \gamma^t}{1 - \gamma} \alpha_2 D_t,$$

which for $t \rightarrow \infty$ simplifies to $\alpha_2/(1 - \gamma)$. Therefore, we have for the average effective overcharge

$$\begin{aligned} \text{plim} \bar{O} &= \text{plim} \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} (p_t - bfp_t) \\ &= \lim \frac{1}{T_E - T_B} \sum_{t=T_B+1}^{T_E} \frac{1 - \gamma^t}{1 - \gamma} \alpha_2 \\ &= \frac{\alpha_2}{1 - \gamma}. \end{aligned}$$

We have for the average estimated overcharge

$$\begin{aligned} \text{plim} \bar{O}_2 &= \text{plim} \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} \left(p_t - \widehat{bfp}_t \right) \\ &= \text{plim} \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} (p_t - bfp_t) - \text{plim} \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} \left(\widehat{bfp}_t - bfp_t \right). \end{aligned}$$

Using Proposition 4 (i) we find that:

$$\begin{aligned} \text{plim} \bar{O}_2 &= \text{plim} \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} (p_t - bfp_t), \quad T_b < T_B, T_e > T_E, \\ &> \text{plim} \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} (p_t - bfp_t), \quad \text{otherwise.} \end{aligned}$$

Combining this with the fact that:

$$\begin{aligned} \text{plim} \frac{1}{T_e - T_b} \sum_{t=T_b+1}^{T_e} (p_t - bfp_t) &= \frac{\alpha_2}{1 - \gamma} \quad T_b > T_B, T_e < T_E, \\ &< \frac{\alpha_2}{1 - \gamma}, \quad \text{otherwise,} \end{aligned}$$

completes the proof. □

Proof of Theorem 3. We have for the effective damage

$$\begin{aligned}
\text{plim } \frac{1}{T} CD &= \text{plim } \frac{1}{T} \sum_{t=T_B+1}^{T_E} (p_t - bfp_t) Q_t \\
&= \text{plim } \frac{1}{T} \sum_{t=T_B+1}^{T_E} \frac{1 - \gamma^t}{1 - \gamma} \alpha_2 Q_t \\
&= \lim \frac{T_E - T_B}{T} \alpha_2 Q_C \lim \frac{1}{T} \sum_{t=T_B+1}^{T_E} \frac{1 - \gamma^t}{1 - \gamma} \\
&= \frac{\alpha_2}{1 - \gamma} (\lambda_E - \lambda_B) Q_C.
\end{aligned}$$

We have for the estimated damage

$$\begin{aligned}
\text{plim } \frac{1}{T} \widehat{CD}_2 &= \text{plim } \frac{1}{T} \sum_{t=T_b+1}^{T_e} (p_t - \widehat{bfp}_t) Q_t \\
&= \text{plim } \frac{1}{T} \sum_{t=T_b+1}^{T_e} (p_t - bfp_t) Q_t + \text{plim } \frac{1}{T} \sum_{t=T_b+1}^{T_e} (bfp_t - \widehat{bfp}_t) Q_t \\
&= \text{plim } \frac{1}{T} CD - \text{plim } \frac{1}{T} \sum_{t=T_b+1}^{T_e} (\widehat{bfp}_t - bfp_t) Q_t.
\end{aligned}$$

We will analyze the limiting behavior of the second term in more detail under Assumption 1. Proposition 4 (i) implies that

$$(\widehat{bfp}_{T_b+s} - bfp_{T_b+s}) Q_{T_b+s} = Q_{T_b+s} \sum_{j=0}^{s-1} (\gamma + \gamma^*)^j v_{T_b+s-j} + O_P(T^{-1/2}),$$

where

$$v_{T_b+s-j} = \gamma^* bfp_{T_b+s-j-1} + \beta^* x_{T_b+s-j} + \alpha_1^* - \varepsilon_{T_b+s-j} + O_P(T^{-1/2}), \quad j = 0, \dots, s-2.$$

We note that

$$bfp_{t-1} = \frac{\alpha_1}{1 - \gamma} + \beta \sum_{s=0}^{\infty} \gamma^s x_{t-1-s} + \sum_{s=0}^{\infty} \gamma^s \varepsilon_{t-1-s}.$$

Due to $m_x = 0$, we can define the linear projection coefficients of a regression of bfp_{t-1} on a constant and x_t as m_p and $\frac{m_{xp-1}}{m_{xx}}$ respectively. Therefore, we can write:

$$\begin{aligned}
bfp_{t-1} &= m_p + \frac{m_{xp-1}}{m_{xx}} x_t + \eta_t, \\
\eta_t &= \beta \sum_{s=0}^{\infty} \gamma^s x_{t-1-s} - \frac{m_{xp-1}}{m_{xx}} x_t + \sum_{s=0}^{\infty} \gamma^s \varepsilon_{t-1-s},
\end{aligned}$$

where by construction $E[\eta_t x_t] = 0$. This leads to

$$E[bfp_{T_b+s-j-1} Q_{T_b+s}] = \frac{m_{xp-1}}{m_{xx}} E[x_{T_b+s-j} Q_{T_b+s}] + \frac{\alpha_1}{1-\gamma} E[Q_{T_b+s}],$$

where we assumed that the projection error η_t is uncorrelated with Q_{T_b+s} .

Using $\gamma^* = m_{xx}c$, $\beta^* = -m_{xp-1}c$ and $\alpha^* = -m_p m_{xx}c$ we get

$$\begin{aligned} E[v_{T_b+s-j} Q_{T_b+s}] &= E[(\gamma^* bfp_{T_b+s-j-1} + \beta^* x_{T_b+s-j} + \alpha_1^* - \varepsilon_{T_b+s-j}) Q_{T_b+s}] + O(T^{-1}) \\ &= m_{xx}c \frac{m_{xp-1}}{m_{xx}} E[x_{T_b+s-j} Q_{T_b+s}] \\ &\quad + m_{xx}c \left(\frac{\alpha}{1-\gamma} - (\lambda_E - \lambda_B) \frac{\alpha_2}{1-\gamma} \right) E[Q_{T_b+s}] \\ &\quad - m_{xp-1}c E[x_{T_b+s-j} Q_{T_b+s}] \\ &\quad - m_p m_{xx}c E[Q_{T_b+s}] \\ &\quad - ((\lambda_e - \lambda_b) \text{plim } \hat{\alpha}_2 - (\lambda_E - \lambda_B) \alpha_2) E[Q_{T_b+s}] \\ &\quad - E[\varepsilon_{T_b+s-j} Q_{T_b+s}] + O(T^{-1}). \end{aligned}$$

Collecting terms, this leads to

$$\begin{aligned} E[v_{T_b+s-j} Q_{T_b+s}] &= E[Q_{T_b+s}] \left\{ (\lambda_E - \lambda_B) \frac{\alpha_2}{1-\gamma} (1 - \text{plim } \hat{\gamma}) - (\lambda_e - \lambda_b) \text{plim } \hat{\alpha}_2 \right\} \\ &\quad - E[\varepsilon_{T_b+s-j} Q_{T_b+s}] + O(T^{-1}). \end{aligned}$$

Using (35) we find:

$$E[v_{T_b+s-j} Q_{T_b+s}] = \begin{cases} -E[\varepsilon_{T_b+s-j} Q_{T_b+s}] + O(T^{-1}), & T_b < T_B, T_e > T_E, \\ -E[\varepsilon_{T_b+s-j} Q_{T_b+s}] + O(1), & \text{otherwise,} \end{cases}$$

where the $O(1)$ remainder term is positive. The difference in estimated and true damage per period is

$$\left(\widehat{bfp}_{T_b+s} - bfp_{T_b+s} \right) Q_{T_b+s} = \sum_{j=0}^{s-1} (\gamma + \gamma^*)^j v_{T_b+s-j} Q_{T_b+s} + O_P(T^{-1/2}),$$

hence

$$\begin{aligned} &\text{plim} \frac{1}{T_e - T_b} \sum_{s=1}^{T_e - T_b} \left(\widehat{bfp}_{T_b+s} - bfp_{T_b+s} \right) Q_{T_b+s} \\ &= \text{plim} \frac{1}{T_e - T_b} \sum_{s=1}^{T_e - T_b} \sum_{j=0}^{s-1} (\gamma + \gamma^*)^j v_{T_b+s-j} Q_{T_b+s} \\ &= - \sum_{j=0}^{s-1} (\gamma + \gamma^*)^j E[\varepsilon_{T_b+s-j} Q_{T_b+s}] \\ &\geq 0, \end{aligned}$$

when $T_b < T_B, T_e > T_E$ and under Assumption 2. In the other cases a strict inequality sign holds. This shows that:

$$\begin{aligned} \text{plim } \frac{1}{T} \widehat{CD}_2 &\leq \text{plim } \frac{1}{T} CD, & T_b < T_B, T_e > T_E \\ &< \text{plim } \frac{1}{T} CD, & \text{otherwise,} \end{aligned}$$

which completes the proof. □

Appendix B: Size Control for DGP (1)

For the Monte Carlo study, data have been generated according to DGP (1) with $\varepsilon_t \sim i.i.n.(0, \sigma_\varepsilon^2)$. Explanatory variables are lagged prices p_{t-1} and, for simplicity, a single cost factor x_t . The cartel dummy is defined as in (2) and set at $T_B = \frac{1}{3}T$ and $T_E = \frac{2}{3}T$. The explanatory variable x_t develops by an AR(1) model

$$x_t = \rho x_{t-1} + v_t, \quad (36)$$

where $v_t \sim i.i.n.(0, \sigma_v^2)$ independent of ε_t , i.e. the cost factor is assumed to be strictly exogenous.

In order to investigate the actual size of various structural break inference procedures, data have been generated under the null hypothesis $H_0 : \alpha_2 = 0$. All experiments have a sample of $T = 100$ observations and the number of replications is 10,000. Without loss of generality we set $\alpha_1 = 100(1 - \gamma)$, so that the average price level in the simulations is 100. We normalized with respect to the variance of the disturbance term σ_ε^2 . We furthermore choose $\gamma = \{0.1, 0.5, 0.9\}$ and $\rho = \{0.1, 0.5, 0.9\}$. These values roughly correspond to a low, intermediate and high degree of serial correlation in the time series p_t and x_t .

To facilitate the comparison of simulation results across experiments, some important design parameters are held fixed. We always set β such that the long-run effect of x on p is unity, i.e. we specify $\beta = 1 - \gamma$. Furthermore, σ_ε^2 is chosen such that the signal-to-noise ratio of the model, defined as

$$\text{SNR} = \frac{\text{Var}(p_t - \varepsilon_t)}{\text{Var}(\varepsilon_t)}, \quad (37)$$

does not change between experiments. Assuming $\sigma_\varepsilon^2 = 1$ for DGP (1), Kiviet (1995) derives the following relation between σ_v^2 and SNR and other model parameters

$$\sigma_v^2 = \frac{1}{\beta^2} \left[\text{SNR} - \frac{\gamma^2}{1 - \gamma^2} \right] \frac{(1 - \gamma^2)(1 - \rho^2)(1 - \gamma\rho)}{1 + \gamma\rho}. \quad (38)$$

We choose $\text{SNR} = 9$ across experiments corresponding with a population R^2 of 0.9.

The correct dynamic specification (1) has been estimated in all experiments, hence OLS estimators are consistent, and analyze actual rejection probabilities of sup F -tests and double maximum (UD and WD) tests proposed by Bai and Perron (1998, 2003, 2006).²⁶ The nominal significance level in the simulations is 5% always. For the sup F -tests, the null hypothesis is no break versus k breaks where we experimented with $k = \{1, 2, 3\}$. The finite sample properties of the various test procedures have been investigated for trimming parameter $\mu = h/T = 0.15$, which implies that a minimum of 15 observations is used in any partition of the data, given that $T = 100$.²⁷

²⁶We used EViews 9 for all calculations.

²⁷We obtained similar size and power properties result for $\mu = 0.05$.

The bootstrap version of the various testing procedures uses a standard non-parametric resampling scheme. First, we obtained the OLS estimator allowing for breaks. Second, a random sample is taken from the empirical distribution of the OLS residuals. Third, the bootstrapped dependent variable is calculated according to equation (1). In the bootstrap scheme, we kept the values of exogenous regressors as before. For the lagged dependent variable regressor, the first observation on the dependent variable is kept also as before and pseudo values for the remaining observations are constructed iteratively. Fourth, we estimated the model and calculated the various test statistics from the resampled data. Repeating steps 2 to 4 of the resampling scheme B times, together with the calculated test statistic on the original data ($B + 1$) generated realizations of the test statistic. A size-corrected test was constructed with these ($B + 1$) realizations by using the appropriate quantile of the bootstrap distribution as critical value. The number of bootstrap replications $B = 199$.

Table 2 shows actual rejection probabilities of asymptotic and bootstrap sup F -tests and double maximum tests. Autoregressive dynamics are varied in both p_t and x_t to analyze their relevance for the accuracy of asymptotic approximations. Asymptotic tests performed well under the null hypothesis when there is only small or moderate autoregressive dynamics in the dependent variable, i.e. $\gamma = \{0.1, 0.5\}$. Rejection frequencies do not exceed 0.10, which is reasonably close to the nominal level of 0.05. However, these tests become oversized when $\gamma = 0.9$. In this case, actual rejection frequencies are also increasing with the persistence in the regressor x . In addition, size distortions of sup F -tests increase with the number of breaks specified under H_1 . In general, size distortions of asymptotic tests are largest when both p and x are highly serially correlated ($\gamma = \rho = 0.9$), corroborating the simulation results for univariate models.²⁸ In contrast, bootstrap tests are always size correct, irrespective of the true value of the autoregressive dynamics. Therefore, in practice one should favour the bootstrap version of the break tests always.

²⁸Diebold and Chen (1996), O'Reilly and Whelan (2005). Unreported simulation results show the same pattern for asymptotic sequential sup F tests.

Table 2: Size of nominal 5% sup F- and double max tests.

γ	ρ	supF(1)	supF(2)	supF(3)	UD max	WD max
asymptotic tests						
0.1	0.1	0.051	0.051	0.047	0.049	0.050
0.1	0.5	0.053	0.052	0.054	0.054	0.049
0.1	0.9	0.055	0.066	0.070	0.064	0.059
0.5	0.1	0.048	0.053	0.057	0.050	0.049
0.5	0.5	0.052	0.057	0.062	0.063	0.055
0.5	0.9	0.057	0.088	0.093	0.071	0.082
0.9	0.1	0.122	0.190	0.263	0.175	0.219
0.9	0.5	0.132	0.220	0.302	0.197	0.238
0.9	0.9	0.195	0.306	0.450	0.293	0.372
bootstrap tests						
0.1	0.1	0.040	0.052	0.040	0.042	0.046
0.1	0.5	0.040	0.048	0.044	0.044	0.046
0.1	0.9	0.038	0.042	0.042	0.034	0.034
0.5	0.1	0.042	0.042	0.042	0.044	0.042
0.5	0.5	0.040	0.046	0.038	0.036	0.034
0.5	0.9	0.040	0.048	0.050	0.042	0.054
0.9	0.1	0.048	0.046	0.050	0.052	0.056
0.9	0.5	0.054	0.056	0.052	0.068	0.058
0.9	0.9	0.052	0.050	0.048	0.048	0.052

Note: $T = 100, \mu = 0.15$.