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Testing for Sphericity in Panels

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Abstract

This manuscript considers locally best invariant tests for sphericity in heterogeneous panel models. The associated test statistics take the form of a weighted sum of individual test statistics, each of which is a quadratic form in a spherical random vector under the null. An exact integral expression for the distribution of such a weighted sum, and hence the null distribution of the test, is provided, valid for any size of the cross-sectional and time series dimensions. Addressing the need to quickly compute approximate p-values in empirical work, a highly accurate saddlepoint approximation is also derived, which offers a substantial improvement over the normal approximation in applications where the cross-sectional dimension is small. A panel stationarity test serves as a numerical example.

Key Words: Saddlepoint Approximation, Panel Data, Locally Best Test.

JEL Classification: C12, C23, C63

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1 Introduction

Testing for spherically symmetric errors in the linear model has a long history. For example, the problems of testing for serial correlation, spatial correlation, unit roots, and stationarity can all be cast in this framework. In most of these problems, no uniformly most powerful test exists, which has lead to the adoption of weaker optimality criteria. The locally best invariant (LBI) test, for example, is that which maximizes the slope of the power function at the null hypothesis among all invariant tests. Invariance, in this context, is with respect to changes in the regression coefficients and scaling of the disturbance vector. In the aforementioned problems, Cliff and Ord's (1973) test for spatial correlation, Dufour and King's (1991) test for serial correlation and unit roots, and Kwiatkowski, Phillips, Schmidt, and Shin's (1992) test for stationarity are all locally best invariant. The statistics associated with these tests are in the form of ratios in quadratic forms in elliptically symmetric random vectors, and, consequently, a sizeable body of literature has emerged dealing with the computation of tail probabilities of such ratios, both exactly (Grad and Solomon, 1955; Imhof, 1961; Davies, 1973; Forchini, 2002) and approximately (Lieberman, 1994; Marsh, 1998; Butler and Paolella, 2008). Forchini's paper is a rich source of references on the topic.

In recent years, there has been considerable interest in the corresponding test procedures for panel data models, see, for example, Baltagi (2005) and the references therein. In this manuscript, we consider an exact locally best invariant test for spherical symmetry in heterogeneous panel models. In the absence of cross-sectional correlation, the test statistic is given by a weighted sum of the corresponding individual test statistics, each of which is in the form of a ratio of quadratic forms in an elliptically symmetric random vector. In the limit as the number of individuals tends to infinity, the null distribution converges to a Gaussian; if, however, one is to control the size of the test in finite samples, then the finite-sample null distribution is required, which, unlike in the pure time-series case, appears to have eluded computation to date.

This paper provides an exact expression for the null distribution of the test. By appealing to a conditioning argument and inverting the corresponding characteristic function, the distribution is obtained in the form of a nested integral. The expression can be evaluated numerically to any desired degree of accuracy and hence used for tabulating critical points. For computing p-values however, as is often required in empirical work, such a calculation is arguably too involved, and an approximation may be desirable. Such an approximation is also presented herein. In particular, we show that the same conditioning argument can be employed to arrive at a saddlepoint approximation. This type of approximation is well-known to offer excellent accuracy in the corresponding testing problems in the simple linear regression model. It is perhaps not surprising that this property carries over to the panel setting to which they are generalized herein.

The remainder of this manuscript is organized as follows. Section 2 summarizes the theory of locally best invariant tests in the panel context. Section 3 provides the exact expression for the corresponding null distributions. The saddlepoint approximation is derived in Section 4, and its accuracy is exemplified by applying it to the panel stationarity test of Hadri and Larsson (2005) in Section 5. Section 6 concludes.

2 Locally Best Tests in Heterogenous Panels

Consider the heterogenous panel data model

$$\mathbf{y}_i = \mathbf{X}_i \boldsymbol{\beta}_i + \mathbf{u}_i, \quad i \in \{1 \ldots N\}, \tag{1}$$

where $\mathbf{X}_i$ is a known constant $T_i \times k_i$ matrix of rank k_i, and the $\mathbf{u}_i$ are mutually independent $T_i \times 1$ random vectors. Each $\mathbf{u}_i$ is distributed according to an elliptically symmetric law, with density

$$f_i(\mathbf{u}_i; \theta, \sigma_i) = \left|\sigma_i^2 \boldsymbol{\Sigma}_i(\theta)\right|^{-1/2} g_i\left(\mathbf{u}_i' \left(\sigma_i^2 \boldsymbol{\Sigma}_i(\theta)\right)^{-1} \mathbf{u}_i\right), \tag{2}$$

for some known function $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ such that f_i is a valid density with respect to Lebesgue measure on $\mathbb{R}_{T_i}$. In (2), each $\boldsymbol{\Sigma}_i(\theta)$ is a known, symmetric, differentiable matrix function of the common, but unknown parameter $\theta \in \Theta \equiv \{\theta \in \mathbb{R} : \boldsymbol{\Sigma}_i > 0 \forall i\}$, and without any loss of generality, we assume that $\boldsymbol{\Sigma}_i(0) = \mathbf{I}_{T_i}$. (Typically, matrices $\boldsymbol{\Sigma}_i(\theta)$ will have the same structure, but possibly different dimensions.) We consider the problem of testing $H_0 : \theta = 0$ against $H_a : \theta = a \in \Theta \cap \mathbb{R}_+$. Our discussion is analogous to that given in King and Hillier (1985) for the simple linear model.

Let

$$\mathbf{M}_i \equiv \begin{cases} \mathbf{I}_{T_i} - \mathbf{X}_i \left(\mathbf{X}_i' \mathbf{X}_i\right)^{-1} \mathbf{X}_i', & \text{if } k_i > 0, \\ \mathbf{I}_{T_i}, & \text{otherwise,} \end{cases}$$

and for each i, choose a matrix $\mathbf{P}_i$ such that $\mathbf{P}_i \mathbf{P}_i' = \mathbf{I}_{T_i - k_i}$ and $\mathbf{P}_i' \mathbf{P}_i = \mathbf{M}_i$. Let $\mathbf{y} = [\mathbf{y}_1', \ldots, \mathbf{y}_N']'$; then the vector $\mathbf{v} = [\mathbf{v}_1', \ldots, \mathbf{v}_N']'$, where $\mathbf{v}_i = \mathbf{P}_i \mathbf{y}_i / \|\mathbf{P}_i \mathbf{y}_i\|$, is a maximal invariant with respect to transformations of the form

$$\mathbf{y} \to \tilde{\mathbf{y}}, \quad \tilde{\mathbf{y}} = [\tilde{\mathbf{y}}_1', \ldots, \tilde{\mathbf{y}}_N']', \quad \tilde{\mathbf{y}}_i = a_i \mathbf{y}_i + \mathbf{X}_i \mathbf{b}_i. \tag{3}$$

The density (with respect to the uniform measure on the unit m_i-sphere) of $\mathbf{v}_i$ under H_θ is

$$h_i(\mathbf{v}_i; \theta) = \left|\mathbf{P}_i \boldsymbol{\Sigma}_i(\theta) \mathbf{P}_i'\right|^{-1/2} \left[\mathbf{v}_i' \left(\mathbf{P}_i \boldsymbol{\Sigma}_i(\theta) \mathbf{P}_i'\right)^{-1} \mathbf{v}_i\right]^{-\frac{m_i}{2}},$$

where $m_i = T_i - k_i$; see Kariya (1980) and King (1980). Thus, from independence, the density of $\mathbf{v}$ is

$$h(\mathbf{v}; \theta) = \prod_{i=1}^{N} h_i(\mathbf{v}_i; \theta), \tag{4}$$

based on which optimal invariant tests can be constructed. From Ferguson (1967, p. 236), a locally best invariant test of size α of $H_0 : \theta = 0$ against $H_a : \theta = a \in \Theta \cap \mathbb{R}_+$ is to reject H_0 whenever

$$\frac{\partial}{\partial \theta} \log h(\mathbf{v}; \theta)\Big|_{\theta=0} > d_1,$$

where d_1 is a constant chosen such that the size of the test is α. Applied to the density (4), this yields critical regions of the form

$$\tau(\mathbf{y}) \equiv \sum_{i=1}^{N} m_i \mathbf{v}_i' \mathbf{P}_i \dot{\mathbf{\Sigma}}_i(0) \mathbf{P}_i' \mathbf{v}_i = \sum_{i=1}^{N} m_i \frac{\mathbf{y}_i' \mathbf{M}_i \dot{\mathbf{\Sigma}}_i(0) \mathbf{M}_i \mathbf{y}_i}{\mathbf{y}_i' \mathbf{M}_i \mathbf{y}_i} > d,$$

where $\dot{\mathbf{\Sigma}}_i(\theta)$ denotes the elementwise derivative of $\mathbf{\Sigma}_i(\theta)$ with respect to θ. The constant d satisfies

$$\begin{aligned} 1-\alpha = \Pr_{\theta=0}[\tau(\mathbf{y}) < d] &= \Pr_{\theta=0}\left[\sum_{i=1}^{N} m_i \frac{\mathbf{u}_i' \mathbf{M}_i \dot{\mathbf{\Sigma}}_i(0) \mathbf{M}_i \mathbf{u}_i}{\mathbf{u}_i' \mathbf{M}_i \mathbf{u}_i} < d\right] \\ &= \Pr_{\theta=0}\left[\sum_{i=1}^{N} m_i \frac{\tilde{\mathbf{u}}_i' \mathbf{A}_i \tilde{\mathbf{u}}_i}{\tilde{\mathbf{u}}_i' \mathbf{I}_{m_i} \tilde{\mathbf{u}}_i} < d\right], \end{aligned} \tag{5}$$

where $\mathbf{A}_i = \mathbf{P}_i \dot{\mathbf{\Sigma}}_i(0) \mathbf{P}_i'$, $\tilde{\mathbf{u}}_i \equiv \mathbf{P}_i \mathbf{u}_i$, and under H_0, $\tilde{\mathbf{u}}_i$, like $\mathbf{u}_i$, has a spherically symmetric density

$$\tilde{f}_i(\tilde{\mathbf{u}}_i) = \tilde{g}_i\left(\tilde{\mathbf{u}}_i' \tilde{\mathbf{u}}_i\right), \tag{6}$$

see Kelker (1970).

3 Exact Null Distribution

In order to determine the critical value d in (5), we require the distribution function of statistics of the form

$$\bar{R} \equiv \frac{1}{N} \sum_{i=1}^{N} R_i, \quad \text{where } R_i \equiv \frac{U_i}{D_i} \equiv \frac{\mathbf{u}_i' \mathbf{A}_i \mathbf{u}_i}{\mathbf{u}_i' \mathbf{u}_i}, \tag{7}$$

each $\mathbf{A}_i$ is a symmetric $m_i \times m_i$ matrix, $\mathbf{u}_i$ has density (6), and, for notational convenience, we write $\mathbf{u}$ and g instead of $\tilde{\mathbf{u}}$ and $\tilde{g}$. The characteristic function of R_i is

$$\psi_{R_i}(t) \equiv \mathbb{E}[\exp\{\mathrm{i}tR_i\}] = {}_1F_1(1/2, m_i/2, \mathrm{i}t\mathbf{A}_i), \tag{8}$$

where $\mathrm{i}^2 \equiv -1$, see, e.g., Hillier (2001). In (8), ${}_1F_1(a, b, \mathbf{Z})$ denotes the confluent hypergeometric function of matrix argument. The fact that this function is notoriously difficult to evaluate with high precision makes it doubtful that the distribution function of $\bar{R}$ could be efficiently obtained by numerical inversion of the characteristic function. When $N = 1$, this difficulty is usually circumvented by exploiting the relation

$$\Pr\left(\frac{\mathbf{u}_1' \mathbf{A}_1 \mathbf{u}_1}{\mathbf{u}_1' \mathbf{u}_1} \leq r\right) = \Pr\left(\mathbf{u}_1' [\mathbf{A}_1 - r\mathbf{I}_{m_1}] \mathbf{u}_1 \leq 0\right)$$

and then using the result of Imhof (1961), but this method fails when $N > 1$. As such, we shall consider a different approach in what follows.

In order to simplify the derivations, suppose for the moment that $m_i = T$ for all i; if $m_i \neq m_j$ for some i, j, all occurrences of T should be replaced by m_i in Theorems 1 and 2; such would be the situation in an unbalanced panel, or if a different number of regressors is included for at least some individuals. Noting that the null distribution of $\bar{R}$ is independent of the specific choice of g_i in (6), we shall further assume, without loss of generality, that

$$g_i(\mathbf{u}_i'\mathbf{u}_i) = (2\pi)^{-T/2} \exp\{\mathbf{u}_i'\mathbf{u}_i\} \quad \forall i,$$

i.e., the $\mathbf{u}_i$ are independently distributed as $\mathrm{N}(\mathbf{0}, \mathbf{I}_T)$.

In order to derive an expression for the distribution of $\bar{R}$ in (7), we will exploit the fact that as a scale-free function of $\mathbf{u}_i$,

$$R_i \perp D_j \quad \forall i, j \in \{1, \ldots, N\},$$

i.e., each ratio R_i in (7) is independent of its own denominator D_i, a result that dates back to Fisher (1930) and Geary (1933). Our use of the result resembles that of Butler and Paolella (1998), who brought it to bear in their derivation of the joint density of the serial correlogram.

Trivially, each R_i is also independent of the denominators of the remaining ratios in the sum, and we have that

$$\begin{aligned}
\Pr\left(\bar{R} \leq \bar{r}\right) &= \Pr\left(\frac{1}{N}\sum_{i=1}^{N} \frac{U_i}{D_i} \leq \bar{r}\right) = \Pr\left(\frac{1}{N}\sum_{i=1}^{N} \frac{U_i}{D_i} \leq \bar{r} \,\middle|\, D_i = 1\ \forall i \in \{1, \ldots, N\}\right) \\
&= \Pr\left(\sum_{i=1}^{N} U_i \leq r \,\middle|\, D_i = 1\ \forall i \in \{1, \ldots, N\}\right),
\end{aligned} \tag{9}$$

where $r = \bar{r}N$.

Next, define $X := \sum_{i=1}^{N} U_i$ and $\mathbf{D} = [D_1, \ldots, D_N]'$, and let $\psi_{X|\mathbf{D}=\mathbf{1}}(s)$ denote the characteristic function (c.f.) of X, conditional on $\mathbf{D} = \mathbf{1}$. By the inversion formula of Gil-Pelaez (1951),

$$\Pr\left(X \leq r \,\middle|\, \mathbf{D} = \mathbf{1}\right) = \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} \mathrm{Im}\left[e^{-\mathrm{i}sr}\psi_{X|\mathbf{D}=\mathbf{1}}(s)\right]\frac{\mathrm{d}s}{s},$$

and it remains to find an expression for the conditional c.f. To that end, first note that from independence, $\psi_{X|\mathbf{D}=\mathbf{1}}(s)$ factors as

$$\psi_{X|\mathbf{D}=\mathbf{1}}(s) = \prod_{i=1}^{N} \psi_{U_i|\mathbf{D}=\mathbf{1}}(s) = \prod_{i=1}^{N} \psi_{U_i|D_i=1}(s).$$

Then, from Bartlett (1938),

$$\psi_{U_i|D_i=1}(s) = \frac{\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \psi_{U_i,D_i}(s,t)e^{-\mathrm{i}t}\mathrm{d}t}{\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \psi_{U_i,D_i}(0,t)e^{-\mathrm{i}t}\mathrm{d}t}, \tag{10}$$

where ψ_{U_i,D_i} is the joint c.f. of U_i and D_i, which is easily seen to be given by

$$\psi_{U_i,D_i}(s) = \prod_{j=1}^{T} \left(1 - 2\mathrm{i}s\omega_{ij} - 2\mathrm{i}t\right)^{-1/2},$$

where $\{\omega_{ij}\}_{j\in\{1,\ldots,T\}}$ are the eigenvalues of $(\mathbf{A}_i + \mathbf{A}_i')/2$. Finally, noting that the denominator in (10) is just the density, at one, of a chi-square with T degrees of freedom, it follows that

$$\psi_{U_i|D_i=1}(s) = \frac{2^{T/2}\Gamma(T/2)e^{1/2}}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}t} \prod_{j=1}^{T} (1 - 2\mathrm{i}s\omega_{ij} - 2\mathrm{i}t)^{-1/2} \, \mathrm{d}t.$$

We have then proven the following result.

Theorem 1. *Let $R_i \equiv \mathbf{u}_i'\mathbf{A}_i\mathbf{u}_i/\mathbf{u}_i'\mathbf{u}_i, i \in \{1,\ldots,N\}$, where the $\mathbf{u}_i$ are independently distributed with densities of the form $g_i(\mathbf{u}_i'\mathbf{u}_i)$. Then the distribution function of $\bar{R} \equiv N^{-1}\sum_{i=1}^N R_i$ can be computed as*

$$F_{\bar{R}}(\bar{r}) = \frac{1}{2} - \frac{1}{\pi}\int_0^{\infty} \mathrm{Im}\left[e^{-\mathrm{i}sN\bar{r}} \prod_{i=1}^{N} \psi_{U_i|D_i=1}(s)\right] \frac{\mathrm{d}s}{s} \tag{11}$$

where

$$\psi_{U_i|D_i=1}(s) = \frac{2^{T/2}\Gamma(T/2)e^{1/2}}{2\pi} \int_{-\infty}^{\infty} e^{-\mathrm{i}t} \prod_{j=1}^{T} (1 - 2\mathrm{i}s\omega_{ij} - 2\mathrm{i}t)^{-1/2} \, \mathrm{d}t,$$

and $\{\omega_{ij}\}_{j\in\{1,\ldots,T\}}$ are the eigenvalues of $(\mathbf{A}_i + \mathbf{A}_i')/2$.

That is, we have expressed the required tail probability as a double integral, which lends itself to numeric evaluation.

4 Saddlepoint Approximation

Even though Theorem 1 provides a computable expression for the critical values of the panel LBI test, its computational complexity may be unacceptably high for routine applications. As such, it will be useful to have available an approximation which improves upon the accuracy of the normal approximation, while at the same time maintaining relative computational simplicity. Such a trade-off is afforded by the saddlepoint approximation. However, because the moment generating function of $\bar{R}$ — as a product of confluent hypergeometric functions of matrix argument — is numerically intractable, it appears infeasible to obtain an asymptotic expansion of its distribution function by a direct application of saddlepoint methods. Instead, the idea is to approximate the conditional distribution appearing in (9) by the double saddlepoint approximation of Skovgaard (1987), which we briefly discuss next.

Let X and $\boldsymbol{Y}$ have dimensions 1×1 and $d \times 1$, respectively, and assume that the random vector $(X, \boldsymbol{Y}')'$ possesses a joint density and a joint cumulant generating function $\mathbb{K}(s, \mathbf{t}) \equiv \log \mathbb{E}[\exp(sX + \mathbf{t}'\boldsymbol{Y})]$. Denote by $\mathbb{K}_{\mathfrak{s}}, \mathfrak{s} \in \{s, \mathbf{t}\}$, the vector of partial derivatives of $\mathbb{K}$ with respect to the elements of $\mathfrak{s}$, and by $\mathbb{K}''(s, \mathbf{t})$ its Hessian. Skovgaard (1987) shows that a saddlepoint approximation for the conditional distribution of X given $\boldsymbol{Y} = \mathbf{y}$ is given by

$$\Pr\left(X \leq x, \boldsymbol{Y} = \mathbf{y}\right) \approx \Phi(\hat{w}) + \phi(\hat{w})\left(\hat{w}^{-1} - \hat{u}^{-1}\right), \tag{12}$$

where

$$\hat{w} \equiv \operatorname{sgn}(\hat{s})\sqrt{2\Big(\hat{s}x + \hat{\mathbf{t}}'\mathbf{y} - K\big(\hat{s},\hat{\mathbf{t}}\big) - \hat{\mathbf{t}}_0'\mathbf{y} + \mathbb{K}(0,\hat{\mathbf{t}}_0)\Big)} \quad \text{and} \quad \hat{u} \equiv \hat{s}\sqrt{\Big|\mathbb{K}''\big(\hat{s},\hat{\mathbf{t}}\big)\Big| \Big/ \Big|\mathbb{K}''\big(0,\hat{\mathbf{t}}_0\big)\Big|}.$$

The quantity $\big(\hat{s},\hat{\mathbf{t}}\big)$ appearing in (12) is commonly referred to as the numerator saddlepoint. It solves the system

$$\begin{aligned}\mathbb{K}_s\big(\hat{s},\hat{\mathbf{t}}\big) &= x \\ \mathbb{K}_{\mathbf{t}}\big(\hat{s},\hat{\mathbf{t}}\big) &= \mathbf{y}.\end{aligned}$$

The denominator saddlepoint $\hat{\mathbf{t}}_0$ solves $\mathbb{K}_{\mathbf{t}}(0,\hat{\mathbf{t}}_0) = \mathbf{y}$. Approximation (12) is the leading term in an asymptotic expansion, the second-order term in which has been derived in Kolassa (1996).

In order to apply this result to the problem at hand, we require the joint cumulant generating function of $X = \sum_{i=1}^N U_i$ and $\mathbf{D} = [D_1,\ldots,D_N]'$, which follows easily from the arguments concerning the joint c.f. above. The result is

$$\begin{aligned}\mathbb{K}(s,\mathbf{t}) &\equiv \log\mathbb{E}\left[\exp\left\{s\sum_{i=1}^N U_i + \sum_{i=1}^N t_i D_i\right\}\right] \\ &= \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^T \log\nu_{ij},\end{aligned}$$

where $\nu_{ij} \equiv (1 - 2s\omega_{ij} - 2t_i)^{-1}$, and for each value of i, $\{\omega_{ij}\}_{j\in\{1,\ldots,T\}}$ are the eigenvalues of $\mathbf{A}_i$. In obvious notation, its derivatives are

$$\begin{aligned}&\mathbb{K}_s(s,\mathbf{t}) = \sum_{i=1}^N\sum_{j=1}^T \omega_{ij}\nu_{ij}, &&\mathbb{K}_{t_i}(s,\mathbf{t}) = \sum_{j=1}^T \nu_{ij}, &&\mathbb{K}_{ss}(s,\mathbf{t}) = 2\sum_{i=1}^N\sum_{j=1}^T \omega_{ij}^2\nu_{ij}^2, \\ &\mathbb{K}_{st_i}(s,\mathbf{t}) = 2\sum_{j=1}^T \omega_{ij}\nu_{ij}^2, &&\mathbb{K}_{t_it_k}(s,\mathbf{t}) = 0,\ i\neq k,\ \text{and} &&\mathbb{K}_{t_it_i}(s,\mathbf{t}) = 2\sum_{j=1}^T \nu_{ij}^2,\end{aligned}$$

so that the numerator saddlepoint solves

$$\begin{aligned} r &= \sum_{i=1}^N\sum_{j=1}^T \omega_{ij}\hat{\nu}_{ij} \\ 1 &= \sum_{j=1}^T \hat{\nu}_{ij}, \quad i\in\{1,\ldots,N\},\end{aligned} \tag{13}$$

where hatted quantities depend on $(\hat{s},\hat{\mathbf{t}})$ rather than $(s,\mathbf{t})$. This system of $N+1$ equations must be solved numerically for each value of $\bar{r}$, which, owing to the sparsity of the problem's Jacobian, is a less daunting task than may at first appear.

The denominator saddlepoint $\hat{\mathbf{t}}_0 = (t_{0,1},\ldots t_{0,N})'$ is given analytically as $\hat{t}_{i,0} = (1-T)/2,\ i \in$

$\{1, \ldots, N\}$, so that

$$\mathbb{K}\big(0, \hat{\mathbf{t}}_0\big) = -(NT/2)\log T \quad \text{and} \quad \Big|\mathbb{K}''\big(0, \hat{\mathbf{t}}_0\big)\Big| = (2/T)^N. \tag{14}$$

A further simplification occurs in (12) by noting that it follows from (13) that

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{T}(\omega_{ij} - \bar{r})\hat{\nu}_{ij} = 0 \Leftrightarrow \quad & (15)\\
\hat{s}\sum_{i=1}^{N}\sum_{j=1}^{T}(\omega_{ij} - \bar{r})\hat{\nu}_{ij} + \sum_{i=1}^{N}\hat{t}_i\sum_{j=1}^{T}\hat{\nu}_{ij} = \hat{s}\cdot 0 + \hat{\mathbf{t}}'\mathbf{1} \Leftrightarrow \quad & \\
\sum_{i=1}^{N}\sum_{j=1}^{T}(\hat{s}\omega_{ij} - \hat{s}\bar{r} + \hat{t}_i)\hat{\nu}_{ij} = \hat{\mathbf{t}}'\mathbf{1} \Leftrightarrow \quad & \\
-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{T}(1 - 2\hat{s}\omega_{ij} - 2\hat{t}_i + 2\hat{s}\bar{r} - 1)\hat{\nu}_{ij} = \hat{\mathbf{t}}'\mathbf{1} \Leftrightarrow \quad & \\
-\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{T}(\hat{\nu}_{ij}^{-1} + 2\hat{s}\bar{r} - 1)\hat{\nu}_{ij} = \hat{\mathbf{t}}'\mathbf{1} \Leftrightarrow \quad & \\
\frac{N(1-T)}{2} = \hat{\mathbf{t}}_0'\mathbf{1} = \hat{s}r + \hat{\mathbf{t}}'\mathbf{1}, \quad & (16)
\end{aligned}$$

so that

$$\hat{w} = \operatorname{sgn}(\hat{s})\sqrt{2\Big(\mathbb{K}\big(0, \hat{\mathbf{t}}_0\big) - K\big(\hat{s}, \hat{\mathbf{t}}\big)\Big)} = \operatorname{sgn}(\hat{s})\sqrt{-\sum_{i=1}^{N}\sum_{j=1}^{T}\log(T\hat{\nu}_{ij})}.$$

Next, we will simplify the expression for $\hat{u}$. In order to economize the notation, let $\hat{\kappa}_{00} \equiv \mathbb{K}_{ss}(\hat{s}, \hat{\mathbf{t}})$, $\hat{\kappa}_{0i} \equiv \mathbb{K}_{st_i}(\hat{s}, \hat{\mathbf{t}}), i \in \{1, \ldots, N\}$, and $\hat{\kappa}_i \equiv \mathbb{K}_{t_it_i}(\hat{s}, \hat{\mathbf{t}}), i \in \{1, \ldots, N\}$. Then differentiating equations (13) and (16) with respect to r gives

$$\frac{\mathrm{d}\hat{s}}{\mathrm{d}r} = \left(\hat{\kappa}_{00} - \sum_{i=1}^{N}\hat{\kappa}_{0i}^2\hat{\kappa}_i^{-1}\right)^{-1}, \quad \frac{\mathrm{d}\hat{t}_i}{\mathrm{d}r} = -\frac{\mathrm{d}\hat{s}}{\mathrm{d}r}\frac{\hat{\kappa}_{0i}}{\hat{\kappa}_i}, \text{ and } \quad 0 = \frac{\mathrm{d}\hat{s}}{\mathrm{d}r}r + \hat{s} + \sum_{i=1}^{N}\frac{\mathrm{d}\hat{t}_i}{\mathrm{d}r},$$

so that

$$
\begin{aligned}
\hat{s}\left(\hat{\kappa}_{00}-\sum_{i=1}^{N}\hat{\kappa}_{0i}^{2}\hat{\kappa}_{i}^{-1}\right) &= -r+\sum_{i=1}^{N}\hat{\kappa}_{0i}\hat{\kappa}_{i}^{-1} \\
&= \hat{s}^{-1}\left(-\hat{s}r+\sum_{i=1}^{N}\frac{2\sum_{j=1}^{T}\hat{s}\omega_{ij}\nu_{ij}^{2}}{2\sum_{j=1}^{T}\nu_{ij}^{2}}\right) \\
&= \hat{s}^{-1}\left(-\hat{s}r+\sum_{i=1}^{N}\frac{1}{2}-\hat{t}_i+\frac{\sum_{j=1}^{T}-\hat{\nu}_{ij}^{2}+2\hat{t}_i\nu_{ij}^{2}+2\hat{s}\omega_{ij}\nu_{ij}^{2}}{2\sum_{j=1}^{T}\nu_{ij}^{2}}\right) \\
&= \hat{s}^{-1}\left(-\hat{s}r+\frac{N}{2}-\left(\frac{N(1-T)}{2}-\hat{s}r\right)+\sum_{i=1}^{N}\frac{-\sum_{j=1}^{T}\hat{\nu}_{ij}}{2\sum_{j=1}^{T}\nu_{ij}^{2}}\right) \\
&= \hat{s}^{-1}\left(\frac{NT}{2}-\sum_{i=1}^{N}\hat{\kappa}_{i}^{-1}\right), \qquad (17)
\end{aligned}
$$

where the penultimate and last equalities follow from (16) and (13), respectively. The determinant of $\mathbb{K}''(\hat{s},\hat{\mathbf{t}})$ appearing in (12) evaluates to

$$
\left|\mathbb{K}''(\hat{s},\hat{\mathbf{t}})\right| = \left(\hat{\kappa}_{00}-\sum_{i=1}^{N}\hat{\kappa}_{0i}^{2}\hat{\kappa}_{i}^{-1}\right)\prod_{i=1}^{N}\hat{\kappa}_i,
$$

which, together with (14) and (17), and with $\hat{\gamma}_i \equiv T\hat{\kappa}_i/2, i \in \{1,\ldots,N\}$, yields

$$
\hat{u} = \operatorname{sgn}(\hat{s})\sqrt{\frac{1}{2}\left[\prod_{i=1}^{N}\hat{\gamma}_i\right]\sum_{i=1}^{N}T(1-\hat{\gamma}_i^{-1})}.
$$

We collect the relevant formulae in the following theorem.

THEOREM 2. *Let $R_i \equiv \mathbf{u}_i'\mathbf{A}_i\mathbf{u}_i/\mathbf{u}_i'\mathbf{u}_i, i \in \{1,\ldots,N\}$, where the $\mathbf{u}_i$ are independently distributed with densities of the form $g_i(\mathbf{u}_i'\mathbf{u}_i)$. Then a saddlepoint approximation to the distribution function of $\bar{R} \equiv N^{-1}\sum_{i=1}^{N}R_i$ is given by*

$$
\hat{F}(\bar{r}) \equiv \Phi(\hat{w})+\phi(\hat{w})\big(\hat{w}^{-1}-\hat{u}^{-1}\big), \qquad (18)
$$

where

$$
\hat{w} = \operatorname{sgn}(\hat{s})\sqrt{-\sum_{i=1}^{N}\sum_{j=1}^{T}\log(T\hat{\nu}_{ij})}, \quad \hat{u} = \operatorname{sgn}(\hat{s})\sqrt{\frac{1}{2}\left[\prod_{i=1}^{N}\hat{\gamma}_i\right]\sum_{i=1}^{N}T(1-\hat{\gamma}_i^{-1})},
$$

$\hat{\nu}_{ij} = (1-2\hat{s}\omega_{ij}-2\hat{t}_i)^{-1}$, $\hat{\gamma}_i = T\sum_{j=1}^{T}\hat{\nu}_{ij}^2$, $\{\omega_{ij}\}_{j\in\{1,\ldots,T\}}$ are the eigenvalues of $(\mathbf{A}_i+\mathbf{A}_i')/2$, and the saddlepoint $(\hat{s},\hat{\mathbf{t}})$ solves

$$
\bar{r} = N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{T}\omega_{ij}\hat{\nu}_{ij}, \quad 1 = \sum_{j=1}^{T}\hat{\nu}_{ij}, \quad i \in \{1,\ldots,N\}.
$$

5 Application

In this section, we demonstrate the accuracy of the saddlepoint approximation by applying it to the stationarity test of Hadri and Larsson (2005). The intuition of the test, which is a generalization of the KPSS test for stationarity (Kwiatkowski et al., 1992) to panel data models, is to decompose the error term, for each individual, into a white noise component and a random walk. The null hypothesis is that the variance of the innovation sequence of the random walk is zero.

The results of this paper allow us to obtain the finite sample null distribution of the test, thus generalizing the results of Hornok and Larsson (2000), who considered the pure time-series case, i.e., the KPSS test. We consider Hadri and Larsson's Model (2), where under the alternative, some series are stationary around incidental trends. In terms of model (1), $\mathbf{X}_i = [\mathbf{1}_{T_i}\ \mathbf{t}_{T_i}]$, where $\mathbf{1}_{T_i}$ is a $T_i \times 1$ vector of ones and $\mathbf{t}_{T_i}$ is a $T_i \times 1$ vector of consecutive natural numbers starting at 1,

$$\sigma_i^2 \boldsymbol{\Sigma}_i(\theta) = \sigma_i^2 \left[\theta \mathbf{F}_{T_i} + \mathbf{I}_{T_i}\right],$$

and the (j,k)th element of $\mathbf{F}_{T_i}$ is $\min(j,k)$. It is immediate that

$$\dot{\boldsymbol{\Sigma}}_i(0) = \mathbf{F}_{T_i},$$

so that the locally best test of $H_0 : \theta = 0$ versus $H_a : \theta > 0$, invariant to transformations of the form (3), is the one which rejects for large values of

$$\tau(\mathbf{y}) = \sum_{i=1}^{N} \tau_i(\mathbf{y}_i) \equiv \sum_{i=1}^{N} (T_i - 2) \frac{\mathbf{y}_i' \mathbf{M}_i \mathbf{F}_{T_i} \mathbf{M}_i \mathbf{y}_i}{\mathbf{y}_i' \mathbf{M}_i \mathbf{y}_i}.$$

In order to ensure that its distribution converge to a standard Gaussian under the null, Hadri and Larsson define their test statistic in the slightly modified form

$$Z_{\tau NT} \equiv \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \frac{\tau_i(\mathbf{y}_i) - (T_i^2 - 4)/15}{\sqrt{(T_i+2)(T_i-2)^2(13T_i^2+23)/(2100T_i) - (T_i^2-4)^2/225}}.$$

Clearly, if $T_i = T \forall i$, then the tests based on $\tau(\mathbf{y})$ and on $Z_{\tau NT}$ are equivalent; if, however, the time series dimensions differ across individuals, then the latter is no longer locally best, but only approximately so. The intuition behind this is as follows: suppose the panel consists of $N-1$ individuals of time series dimension T_1, and one — individual N, say — with time series dimension T_N, much larger than T_1. Suppose that the Nth series contains a unit root, which is correctly detected by the LBI test. The test based on $Z_{\tau NT}$ places a relatively lower weight on the Nth individual statistic, and, hence, may not detect the nonstationarity.

Figure 1 offers a comparison of the approximate null distributions of the $Z_{\tau NT}$-test obtained from the saddlepoint and normal approximations, respectively, for a model with $N = 10$ and $T_i = 10$, $i \in \{1, \ldots, N\}$. Depicted is the relative error in percent, defined as $100(\hat{F} - F)/\min(F, 1-F)$, where $\hat{F}$ denotes the respective approximate distribution function, and the exact values F have

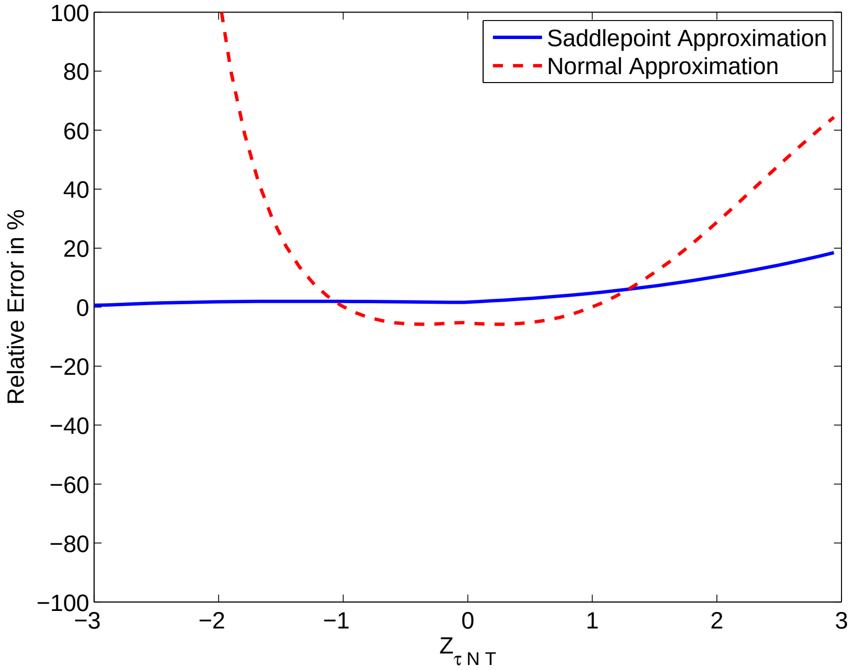

Figure 1: Relative Error in %, defined as $100(\hat{F}(x) - F(x))/\min(F(x), 1 - F(x))$, of approximations for the distribution of $Z_{\tau NT}$, where $N = T = 10$.

been computed from (11). The computation of the 55 tail probabilities in the graph took 429 milliseconds using the saddlepoint approximation, whereas the exact values required 43.4 seconds, about 100 times longer. As expected, the normal approximation deteriorates in the tails of the distribution — the crucial part of the support in hypothesis testing —, whereas the relative error of the saddlepoint approximation never exceeds 20%. This is in stark contrast with the Cornish-Fisher expansion explored by Hadri and Larsson, who note on Page 60:

> We tried to improve the empirical size of our two tests particularly for $T < 10$ by using the Fisher-Cornish expansion [...]. However, the improvements were very marginal. This is not surprising as it is well known that the Fisher-Cornish expansion like the Edgeworth expansion from which it is derived deteriorates in the tails.

The saddlepoint approximation, on the other hand, can be thought of as an Edgeworth expansion applied to the exponentially tilted density, and does not suffer from this deficiency.

6 Conclusions

The locally best invariant test for sphericity in a heterogeneous panel model is given by a weighted sum of the individual tests. While the limiting (as $N \to \infty$) null distribution of the appropriately scaled test statistic is Gaussian, the finite sample distribution is of considerable interest, for the following reasons: First, in macroeconomic panels, the cross-section dimension is typically small, so that use of asymptotic critical values will incur significant size distortions; second, even for moderate values of N, the quality of the normal approximation, for any particular test, is unknown a priori. The results provided herein address both of these problems: the exact integral expression for the distribution function is useful for determining the accuracy of the Normal approximation when devising a test; the saddlepoint approximation, owing to its relative ease of computation, can serve as a routine tool for the computation of p-values. While also approximate in nature, the saddlepoint p-values have relative error, as opposed to the absolute error associated with the normal approximation, and thus preserve good accuracy even in the extreme tails of the distribution.

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